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The Complete Convergence Theorem Holds for Contact Processes on Open Clusters of $\mathbb{Z}^d \times \mathbb{Z}^+$

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Abstract We study contact processes on open clusters of half space. Our result shows that the complete convergence theorem holds.

Keywords Contact process · Percolation · Open cluster · Half plane · Complete convergence theorem

1 Introduction

Set $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$. Denote $\mathbb{H} := \mathbb{Z}^d \times \mathbb{Z}^+$ and $\mathbb{E} := \{(x, y) : \|x - y\| = 1, x \in \mathbb{H}, y \in \mathbb{H}\}$. That is, $(\mathbb{H}, \mathbb{E})$ is the integer lattice of the half space. Run the Bernoulli bond percolation model on the lattice in the way that each edge is declared open with probability p and closed with probability $1 - p$, where $0 < p \leq 1$. Different edges receive independent declarations. Delete all closed edges and get a graph $\mathcal{C}$. Call each connected component of $\mathcal{C}$ an open cluster. More formally, we consider the following probability space. As sample space we take $\Omega^b = \{0, 1\}^{\mathbb{E}}$, points of which are represented as $\omega = (\omega(e) : e \in \mathbb{E})$. The value $\omega(e) = 0$ corresponds to e being closed, and $\omega(e) = 1$ corresponds to e being open. We take $\mathcal{F}^b$ to be the σ-field of subsets of Ω^b generated by the finite-dimensional cylinders. Finally, we take product measure with density p on $(\Omega^b, \mathcal{F}^b)$; this is the measure $\mathbf{P}^p = \prod_{e \in \mathbb{E}} \mu_e$, where μ_e is Bernoulli measure on $\{0, 1\}$, given by

$$\mu_e(\omega(e) = 1) = p, \qquad \mu_e(\omega(e) = 0) = 1 - p.$$

Then

$$\mathcal{C} = \mathcal{C}(\omega) = (\mathbb{H}, \{e \in \mathbb{E} : \omega(e) = 1\}).$$

Readers can refer to Grimmett [6] for more background on percolation.

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Then run the contact process on the random graph $\mathcal{C}$. The contact process, introduced by Harris [7], is a continuous-time Markov process whose state space is the set of all subsets of $\mathbb{H}$. It may be thought of as a model for the spread of infection. Here is a heuristic description. At each instant of (continuous) time, each site is in one of two states: infected or healthy. An infected site recovers at rate 1 and a healthy site becomes infected at rate proportional to the number of its infected neighbors. Formally, $\xi = \xi(\mathcal{C}) = \{\xi_t(\mathcal{C}): t \geq 0\}$ is the contact process on $\mathcal{C}$ with transition rates

$$\begin{cases} \xi_t \to \xi_t \setminus \{x\} & \text{for } x \in \xi_t \text{ at rate } 1, \\ \xi_t \to \xi_t \cup \{x\} & \text{for } x \notin \xi_t \text{ at rate } \lambda \cdot |\{y \in \xi_t : (y,x) \text{ is open}\}|, \end{cases}$$

where λ is a parameter, and $|\cdot|$ denotes the cardinality of a set. Denote by $\xi^A(\mathcal{C})$ the process with initial state A. Realize $\xi(\mathcal{C})$ in the probability space $(\Omega^s, \mathcal{F}^s, \mathbf{P}_\lambda)$. We say ξ^A survives if $\xi_t^A \neq \emptyset$ for all $t \geq 0$, while ξ^A dies out if there exists $t > 0$ such that $\xi_t^A = \emptyset$.

If $p = 1$, then our model reduces to the contact process on the half space. If $p < 1$, then $\mathcal{C}$ is a random graph. Hence our model is a kind of contact process in a random environment. It is a special case of Klein [9] with $\delta \equiv 1$ and λ being a Bernoulli random variable. Readers can refer to Bramson, Durrett & Schonmann [2], Klein [9], Pemantle & Stacey [11] and Steif & Warfheimer [13] for more information about contact process in a random environment. For more general surveys on the theory of contact process, we refer the readers to Bezuidenhout & Grimmett [1], Durrett [3, 4], Griffeath [5] and Liggett [10].

It is well known that the complete convergence theorem holds for the contact process on $\mathbb{Z}^d$, see Bezuidenhout & Grimmett [1]. Now if $\mathbf{P}_\lambda(\xi^0(\mathcal{C})$ survives$) > 0$, does $(\xi_t^0(\mathcal{C}), t \geq 0)$ have a limit distribution? The answer is yes, and we can verify that the complete convergence theorem still holds. Let p^* be the critical value of the Bernoulli bond percolation model on $\mathbb{H}$. When $p \leq p^*$ there is no infinite open cluster for almost ω, see Sect. 7.3 of Grimmett [6]. So $\xi_t^{\mathbb{H}}(\mathcal{C})$ converges weakly to $\delta_\emptyset$ for each λ, where $\delta_\emptyset$ is the probability measure putting mass one on $\emptyset$. When $p > p^*$ there exists $\mathcal{C}_\infty$, a unique infinite open cluster, of $\mathcal{C}$ for almost ω. Denote by $\nu_{\mathcal{C}}$ the upper invariant measure, that is, the weak limit of the distribution of $\xi_t^{\mathbb{H}}(\mathcal{C})$ as $t \to \infty$. We have the following complete convergence theorem.

Theorem 1.1 *For $p^* < p \leq 1$ and $\lambda > 0$, there exists $\Omega_0 \subseteq \Omega^b$ with $\mathbf{P}^p(\Omega_0) = 1$, such that for all $\omega \in \Omega_0$ and $A \subseteq \mathbb{H}$,*

$$\xi_t^A(\mathcal{C}) \quad \Rightarrow \quad \nu_{\mathcal{C}} \cdot \mathbf{P}_\lambda(\xi^{A\cap\mathcal{C}_\infty}(\mathcal{C}) \text{ survives}) + \delta_\emptyset \cdot \mathbf{P}_\lambda(\xi^{A\cap\mathcal{C}_\infty}(\mathcal{C}) \text{ dies out})$$

as t tends to infinity, where '$\Rightarrow$' stands for weak convergence.

Remark Our result applies also to contact processes on open clusters of Bernoulli site percolation.

2 Outline of Proof

Our work is enlightened by Bezuidenhout & Grimmett [1]. They construct a space-time box $B = \lceil -L, L\rfloor^d \times [0,t]$ such that with high probability, a seed on the bottom is joined within B to another seed on each of the other $2d+1$ faces of B. However, we construct several types of space boxes. It is hard for us to describe the construction in simple words. Since $\mathcal{C}$

is not transitive, we have to be more careful on the construction of boxes in order to apply 'translation invariance' of $\mathcal{C}$.

When $p = 1$, our result can be directly deduced by the argument of Bezuidenhout & Grimmett [1]. In this paper we only prove the case $d = 1$ and $p < 1$. Our technique still works for higher dimensions $d \geq 2$. When $d = 1$, the lattice $(\mathbb{H}, \mathbb{E})$ can be embedded into the complex plane. Hence we reset $\mathbb{H} = \{a + b\mathrm{i} : a \in \mathbb{Z}, b \in \mathbb{Z}^+\}$ and $\mathbb{E} = \{(x, y) : x, y \in \mathbb{H}, |x - y| = 1\}$, where $\mathrm{i} = \sqrt{-1}$ is the unit imaginary number. Denote by $\mathbf{R}^+$ the set of nonnegative numbers and $\overline{\mathbf{R}^+} = \mathbf{R}^+ \cup \{\infty\}$. For a real number a, let $[a]$ be the largest integer which is no greater than a. For a complex number x, denote by $\Re(x)$ its real part, and by $\Im(x)$ its imaginary part. Let

$$\begin{aligned} \lceil a, b \rfloor := \{x \in \mathbb{H} : \ & \min\{\Re(a), \Re(b)\} \leq \Re(x) \leq \max\{\Re(a), \Re(b)\}, \\ & \min\{\Im(a), \Im(b)\} \leq \Im(x) \leq \max\{\Im(a), \Im(b)\}\}. \end{aligned}$$

That is, $\lceil a, b \rfloor$ is a site set, which forms the rectangle in $\mathbb{H}$ with diagonal sites a and b. Set $B_x(M) = \lceil x - M - M\mathrm{i}, x + M + M\mathrm{i} \rfloor \cap \mathbb{H}$ for $x \in \mathbb{H}$ and $M \in \mathbb{Z}^+$.

We shall make abundant use of the graphical representation of the contact process due to Harris [8]. We follow the notation of Bezuidenhout & Grimmett [1]. Fix $\mathcal{C}$ and think of the process as being imbedded in space-time. Along each 'time-line' $x \times [0, \infty)$ are positioned 'deaths' at the points of a Poisson process with intensity 1. For each open edge (x_1, x_2) of $\mathcal{C}$, between $x_1 \times [0, \infty)$ and $x_2 \times [0, \infty)$ are positioned edges directed from the first to the second having centers forming a Poisson process of intensity λ on the set $\frac{1}{2}(x_1 + x_2) \times [0, \infty)$. These Poisson processes are taken to be independent of one another. The random graph obtained from $\mathbb{H} \times [0, \infty)$ by deleting all points at which a death occurs and adding in all directed edges can be used as a percolation superstructure on which a realization of the contact process is built. We shall make free use of the language of percolation. For example, for $A, B \subseteq \mathbb{H} \times [0, \infty)$, we say that A is joined to B if there exist $a \in A$ and $b \in B$ such that there exists a path from a to b traversing time-lines in the direction of increasing time (but crossing no death) and directed edges between such lines; for $C \subseteq \mathbb{H} \times [0, \infty)$, we say that A is joined to B within C if such a path exists using segments of time-lines lying entirely in C. But we extend the notion 'within' in this paper. For $A, B \subseteq \mathbb{H} \times [0, \infty)$ and $C \subseteq \mathbb{H}$, we say that A is joined to B within C if such a path exists using segments of time-lines lying entirely in $C \times [0, \infty)$; for $D \subseteq \mathbb{E}$, we say that A is joined to B within D if such path exists using directed edges having centers lying entirely in $D' \times [0, \infty)$, where $D' = \{\frac{x_1 + x_2}{2} : (x_1, x_2) \in D\}$.

For $x \in \mathbb{H}$, $r \in \mathbb{Z}^+$ and $t \in [0, \infty)$, we call $(x \times t)_r$ a horizontal (resp. vertical) seed with $2r + 1$ sites if all sites in $\lceil x - r, x + r \rfloor$ (resp. $\lceil x - r\mathrm{i}, x + r\mathrm{i} \rfloor$) are infected at time t. The word 'seed' comes from Grimmett [6]. We say that a horizontal seed $(x \times s)_r$ is joined to a vertical seed $(y \times t)_r$ if $\lceil x - r, x + r \rfloor \times s$ is joined to $z \times t$ for all $z \in \lceil y - r\mathrm{i}, y + r\mathrm{i} \rfloor$.

Denote by $\mathbf{P}_\lambda^p$ a probability measure which satisfies

$$\mathbf{P}_\lambda^p(\cdot) = \int \mathbf{P}_\lambda(\xi(\mathcal{C}) \in \cdot)\mathbf{P}^p(d\omega).$$

Generally, $\mathbf{P}_\lambda^p$ is called the annealed law, $\mathbf{P}_\lambda$ the quenched law. We use the annealed law in Sects. 3 and 4 to find different kinds of boxes. In Sect. 5, we use the quenched law to prove Theorem 1.1 by the result of the annealed law.

Suppose $p < 1$ and $\mathbf{P}_\lambda^p(\xi^0 \text{ survives}) > 0$. In Sect. 3, we find large r, such that $\xi^{\lceil -r, r \rfloor}$ survives with large $\mathbf{P}_\lambda^p$-probability. Next we find two kinds of edge sets: S-boxes and L-boxes. These edge sets are called boxes since the endpoints of each edge set form a rec-

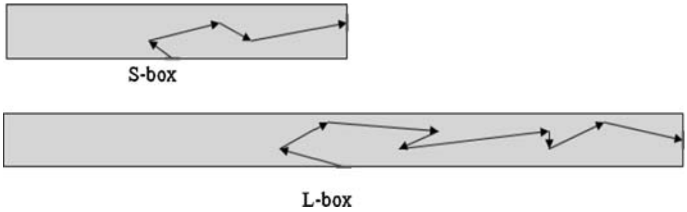
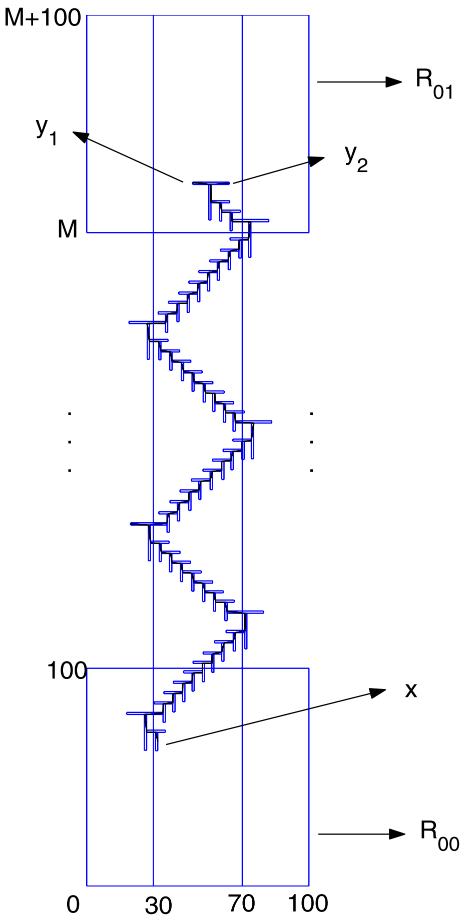


Fig. 1 S-box and L-box

Fig. 2 Sample of $\mathring{A}_1^v(s, x, \mathrm{i})$



tangle on $\mathbb{H}$. We show that with large probability, a horizontal seed on the bottom of each kind of box is joined to a vertical seed on the right side within the box. See Fig. 1.

In Sect. 4, we first use S-boxes and L-boxes to construct a route (through Algorithm 1) so that with large probability, a seed in a fixed square is joined by the route to seeds in each of the other two fixed squares (one above, the other on the right), see Figs. 2 and 3. Next, we use the renormalization method to construct integrated boxes (through Algorithm 2). Consider each of the fixed squares as one single point and connect two adjacent points with an ori-

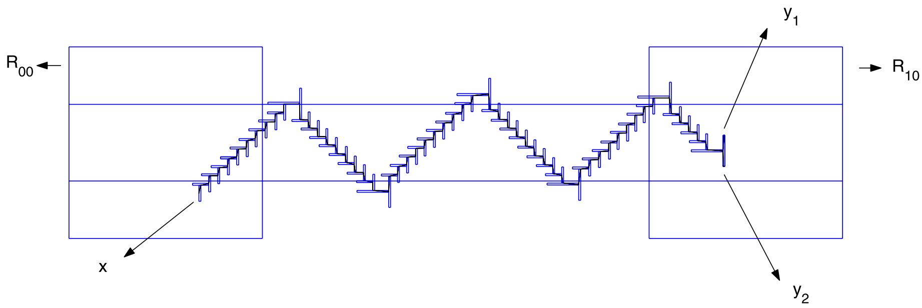
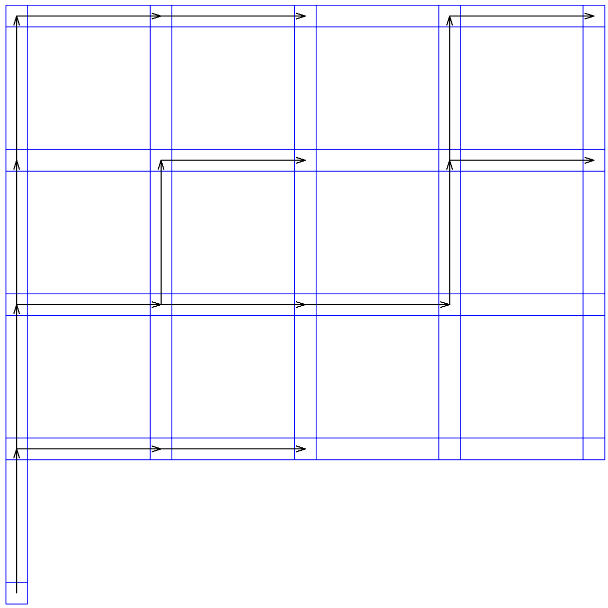


Fig. 3 Sample of $\mathring{A}_1^{v}(s, x, 1)$

Fig. 4 Renormalization



ented edge if the seed in the first square is joined to a seed in the second square through Algorithm 1. See Fig. 4 for intuition. We can get that with large probability, a seed in $\lceil a, b \rfloor \times 0$ is joined within integrated boxes to another seed in $\lceil a + cn, b + cn \rfloor \times [7\overline{W}n/6, 11\overline{W}n/6]$ for any large n. See Proposition 4.1 for details. The proof of Proposition 4.1 is intuitive but somewhat cumbersome, and is deferred to Appendix 2. The idea is to couple with the oriented site percolation and to transform our problem to the calculation the sum of a sequence of independent random variables.

Theorem 1.12 of Liggett [10] says that on a locally finite connected graph $\mathbb{G}$, the complete convergence theorem holds if and only if the following two assertions hold:

(i) $\mathbf{P}_\lambda(x \in \limsup \xi_t^A(\mathbb{G})) = \mathbf{P}_\lambda(\xi^A(\mathbb{G}) \text{ survives})$ for all $x \in \mathbb{G}$ and $A \subset \mathbb{G}$;

(ii) $\lim_{l\to\infty} \liminf_{t\to\infty} \mathbf{P}_\lambda(\xi_t^{B_0(l)}(\mathbb{G}) \cap B_0(l) \neq \emptyset) = 1$.

In Sect. 5, we check (i) and (ii) for almost all ω. Simply speaking, we iterate the above construction of integrated boxes four times to get that with large probability, a seed in

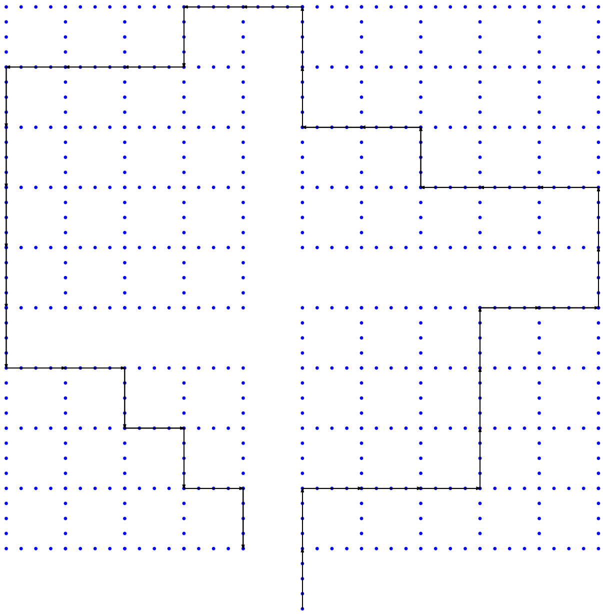


Fig. 5 Strong survival

$\lceil a, b \rfloor \times 0$ is joined to another seed in $\lceil e, f \rfloor \times [3\overline{W}n, \infty)$. See Fig. 5. From this, we get (i). Extra tricks are needed to check (ii) , which are given in Algorithms 3 and 4. They are similar to Algorithms 1 and 2. Therefore, we can get that for each n, with large probability a seed in $\lceil \tilde{a}, \tilde{b} \rfloor \times [0, \tilde{W}]$ is joined to another seed in $\lceil \tilde{e}, \tilde{f} \rfloor \times [(n-1)\tilde{W}, (n+1)\tilde{W}]$. Together with the fact that every remote site cannot be infected in a short time, we get (ii).

Finally, we show some relationship with the contact process on open clusters of the whole plane in Sect. 6.

3 Construction of S-Boxes and L-Boxes

In this section our aim is to find two kinds of edge sets, S-boxes and L-boxes. We will prove that with large $\mathbf{P}_\lambda^p$-probability, a horizontal seed on the bottom of each box is joined within the box to a vertical seed on the right side. See Fig. 1.

Lemma 3.1 *Suppose $\mathbf{P}_\lambda^p(\xi^0$ survives$) > 0$ and $\varepsilon > 0$. Then there exists a positive integer r, such that*

$$\mathbf{P}_\lambda^p(\xi^{\lceil -r,r \rfloor} \text{ survives}) > 1 - \frac{\varepsilon^2}{3}. \tag{3.1}$$

Proof Since $\mathbf{P}_\lambda^p(\xi^0$ survives$) > 0$, then

$$\lim_{M\to\infty} \mathbf{P}_\lambda^p(\forall t,\ \xi_t^0 \subset B_0(M)) = \mathbf{P}_\lambda^p(\xi^0 \text{ dies out}) < 1. \tag{3.2}$$

For fixed M, n (which will be specified later), take $x_1 < x_2 < \cdots < x_n$ such that $x_{k+1} - x_k \geq 3M$ for $k = 1, 2, \ldots, n-1$. Then

$$\begin{aligned}&\mathbf{P}^p_\lambda(\xi^{\{x_1,\ldots,x_n\}} \text{ dies out})\\&\quad\le \mathbf{P}^p_\lambda(\forall i\ \forall t,\ \xi_t^{x_i}\subset B_{x_i}(M))+\mathbf{P}^p_\lambda(\exists i\ \exists t,\ \xi_t^{x_i}\not\subset B_{x_i}(M) \text{ and } \xi^{x_i} \text{ dies out}).\end{aligned}$$

Note that by our choice of $x_1,\ldots,x_n$, the events $\{\forall t,\ \xi_t^{x_i}\subset B_{x_i}(M)\}(i=1,2,\ldots,n)$ are independent since $B_{x_1}(M+1),\ldots,B_{x_n}(M+1)$ are disjoint. Together with the translation invariance, we have

$$\begin{aligned}\mathbf{P}^p_\lambda(\forall i\ \forall t,\ \xi_t^{x_i}\subset B_{x_i}(M)) &= \prod_{i=1}^n \mathbf{P}^p_\lambda(\forall t,\ \xi_t^{x_i}\subset B_{x_i}(M))\\&=[\mathbf{P}^p_\lambda(\forall t,\ \xi_t^0\subset B_0(M))]^n\\&\le [\mathbf{P}^p_\lambda(\xi^0 \text{ dies out})]^n.\end{aligned}$$

For $\varepsilon>0$, take n_0 such that $[\mathbf{P}^p_\lambda(\xi^0 \text{ dies out})]^{n_0}<\varepsilon^2/6$. Also, by the translation invariance, we have

$$\begin{aligned}\mathbf{P}^p_\lambda(\exists i\ \exists t,\ \xi_t^{x_i}\not\subset B_{x_i}(M) \text{ and } \xi^{x_i} \text{ dies out}) &\le n\cdot \mathbf{P}^p_\lambda(\exists t,\ \xi_t^0\not\subset B_0(M) \text{ and } \xi^0 \text{ dies out})\\&= n\cdot[\mathbf{P}^p_\lambda(\xi^0 \text{ dies out})-\mathbf{P}^p_\lambda(\forall t,\ \xi_t^0\subset B_0(M))].\end{aligned}$$

Furthermore, by (3.2) we can take M such that

$$\mathbf{P}^p_\lambda(\xi^0 \text{ dies out})-\mathbf{P}^p_\lambda(\forall t,\ \xi_t^0\subset B_0(M))<\frac{\varepsilon^2}{6n_0}.$$

Together,

$$\mathbf{P}^p_\lambda(\xi^{\{x_1,\ldots,x_{n_0}\}} \text{ dies out})<\frac{\varepsilon^2}{3}.$$

Therefore, if we take r large enough such that $\lceil -r,r\rfloor \supseteq \{x_1,\ldots,x_{n_0}\}$, then by the monotonicity of the contact process, $\mathbf{P}^p_\lambda(\xi^{\lceil -r,r\rfloor} \text{ survives})>1-\varepsilon^2/3$, as desired. □

Fix $\varepsilon>0$. Fix $r\ge 1$ which satisfies (3.1). For $h\ge 40000r$ and $w\ge 40000r$, define random set

$$\begin{aligned}&\Phi^L(h,w)\\&\quad:=\{x\in\lceil -w,-w+hi\rfloor:\ \lceil -r,r\rfloor\times 0 \text{ is joined to } x\times[0,\infty) \text{ within } \lceil -w,w+hi\rfloor\}.\end{aligned}$$

Hence $\Phi^L(h,w)$ is a subset of the left side of the box $\lceil -w,w+hi\rfloor$. Similarly define $\Phi^R(h,w)$ the subset of the right side. Define random set $\Phi^{UL}(h,w)$ the subset of the left part of the up side as follows:

$$\begin{aligned}&\Phi^{UL}(h,w)\\&\quad:=\{x\in\lceil -w+hi,hi\rfloor:\ \lceil -r,r\rfloor\times 0 \text{ is joined to } x\times[0,\infty) \text{ within } \lceil -w,w+hi\rfloor\}.\end{aligned}$$

Similarly define $\Phi^{UR}(h,w)$ the subset of the right part. Set $\Phi^U(h,w):=\Phi^{UL}(h,w)\cup\Phi^{UR}(h,w)$.

Lemma 3.2 *Let $p < 1$ and $\mathbf{P}_\lambda^p(\xi^0$ survives$) > 0$. For any $\varepsilon, N > 0$, one of the following two assertions must be true*:
(1) *There exist constants h, w with $w = 4h$, such that*

$$\mathbf{P}_\lambda^p(|\Phi^L(h,w)| > N) > 1-\varepsilon, \qquad \mathbf{P}_\lambda^p(|\Phi^L(h,2w)| > N) > 1-\varepsilon. \tag{3.3}$$

(2) *There exist constants h, w with $8h \ge w$, such that*

$$\mathbf{P}_\lambda^p(|\Phi^{UL}(h,w)| > N) > 1-\varepsilon, \qquad \mathbf{P}_\lambda^p(|\Phi^L(2h,w)| > N) > 1-\varepsilon. \tag{3.4}$$

Proof Take $w_n = 2^n$ and $h_n = 2^{w_n^2}$. There are two properties of the sequence of boxes $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$. First, all sites being joined with $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$ are contained in $\lceil -w_{n+1}, w_{n+1} + h_{n+1}\mathrm{i}\rfloor$. Second, with large probability there are no edges being joined with $\lceil -w_n + h\mathrm{i}, w_n + h\mathrm{i}\rfloor$ in $\mathcal{C}$ for some $1 < h < h_n - 1$ if n is large enough. By the second property, there exists n_0 such that

$$\mathbf{P}_\lambda^p(\Phi^U(h_n, w_n) = \emptyset) > 1 - \frac{\varepsilon^2}{3} \tag{3.5}$$

for $n > n_0$. If all the edges outside the box $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$ and joined with $\Phi^L(h_n,w_n) \cup \Phi^R(h_n,w_n) \cup \Phi^U(h_n,w_n)$ are closed, then $\xi^{\lceil -r,r\rfloor}$ dies out at a finite time, since there are no infected sites outside $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$. It implies that

$$\mathbf{P}_\lambda^p\big(\xi^{\lceil -r,r\rfloor} \text{ dies out} \mid |\Phi^L(h_n,w_n) \cup \Phi^R(h_n,w_n) \cup \Phi^U(h_n,w_n)| \le 2N\big) \ge (1-p)^{2N+2}.$$

By the first property of the sequence and the strong Markov property,

$$\mathbf{P}_\lambda^p(\exists n_1, \forall n > n_1, |\Phi^L(h_n,w_n) \cup \Phi^R(h_n,w_n) \cup \Phi^U(h_n,w_n)| > 2N \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) = 1.$$

Hence there exists $n_1 > n_0$ such that for $n > n_1$,

$$\mathbf{P}_\lambda^p(|\Phi^L(h_n,w_n) \cup \Phi^R(h_n,w_n) \cup \Phi^U(h_n,w_n)| > 2N \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) > 1 - \frac{\varepsilon^2}{3}. \tag{3.6}$$

By (3.1), (3.5) and (3.6),

$$\mathbf{P}_\lambda^p(|\Phi^L(h_n,w_n) \cup \Phi^R(h_n,w_n)| > 2N) > 1 - \varepsilon^2.$$

Using the FKG inequality (see Theorem 2.4 of Grimmett [6]), we can get

$$\begin{aligned}\varepsilon^2 &\ge \mathbf{P}_\lambda^p(|\Phi^L(h_n,w_n) \cup \Phi^R(h_n,w_n)| \le 2N)\\ &\ge \mathbf{P}_\lambda^p(|\Phi^L(h_n,w_n)| \le N, |\Phi^R(h_n,w_n)| \le N)\\ &\ge \mathbf{P}_\lambda^p(|\Phi^L(h_n,w_n)| \le N)^2.\end{aligned}$$

Consequently,

$$\mathbf{P}_\lambda^p(|\Phi^L(h_n,w_n)| > N) > 1-\varepsilon. \tag{3.7}$$

Comparing (3.7) with (3.3), we see that h_n is much larger than what we want. Hence we reduce the height (such as $h_n/2^k$). Let $k_n' = w_n^2 - n + 2$ and $h_n' = h_n/2^{k_n'}$ for all n. Then $4h_n' = w_n$. If

$$\mathbf{P}_\lambda^p(|\Phi^L(h_n',w_n)| > N) > 1-\varepsilon, \qquad \mathbf{P}_\lambda^p(|\Phi^L(h_n',2w_n)| > N) > 1-\varepsilon$$

for some n, then (1) is true. Otherwise, one of the two following statements must be true:

(3) There exists a subsequence (n_i) such that $\mathbf{P}_\lambda^p(|\Phi^L(h'_{n_i}, w_{n_i})| > N) \le 1 - \varepsilon$.

(4) There exists a subsequence (n_i) such that $\mathbf{P}_\lambda^p(|\Phi^L(h'_{n_i}, 2w_{n_i})| > N) \le 1 - \varepsilon$.
No matter which of the two statements is true, there exists a subsequence (n_i) such that $\mathbf{P}_\lambda^p(|\Phi^L(h'_{n_i}, w'_{n_i})| > N) \le 1 - \varepsilon$ with $8h'_{n_i} \ge w'_{n_i}$. As a result, there exists $0 \le k \le k'_{n_i}$ such that

$$8h_i^* \ge w_i^*, \quad \mathbf{P}_\lambda^p(|\Phi^L(2h_i^*, w_i^*)| > N) > 1 - \varepsilon \quad \text{and} \quad \mathbf{P}_\lambda^p(|\Phi^L(h_i^*, w_i^*)| > N) \le 1 - \varepsilon, \tag{3.8}$$

where $h_i^* = h_{n_i}/2^{k+1}$, $w_i^* = w'_{n_i}$.

We conclude that

$$\mathbf{P}_\lambda^p(|\Phi^U(h_{i^*}^*, w_{i^*}^*)| > 2N) > 1 - \varepsilon^2 \tag{3.9}$$

for some i^*. In fact, if no such i^* exists, then $\mathbf{P}_\lambda^p(|\Phi^U(h_i^*, w_i^*)| > 2N) \le 1 - \varepsilon^2$ for all i. Use the FKG inequality again,

$$\begin{aligned}
&\mathbf{P}_\lambda^p(|\Phi^L(h_i^*, w_i^*) \cup \Phi^R(h_i^*, w_i^*) \cup \Phi^U(h_i^*, w_i^*)| \le 6N)\\
&\quad \ge \mathbf{P}_\lambda^p(|\Phi^L(h_i^*, w_i^*)| \le 2N, |\Phi^R(h_i^*, w_i^*)| \le 2N, |\Phi^U(h_i^*, w_i^*)| \le 2N)\\
&\quad \ge \mathbf{P}_\lambda^p(|\Phi^L(h_i^*, w_i^*)| \le 2N)\mathbf{P}_\lambda^p(|\Phi^R(h_i^*, w_i^*)| \le 2N)\mathbf{P}_\lambda^p(|\Phi^U(h_i^*, w_i^*)| \le 2N) \ge \varepsilon^4.
\end{aligned}$$

However, h_i^* tends to infinity as $i \to \infty$, which implies that there exists a strictly increasing subsequence $(h_{i_j}^*)$ such that

$$\mathbf{P}_\lambda^p(|\Phi^L(h_{i_j}^*, w_{i_j}^*) \cup \Phi^R(h_{i_j}^*, w_{i_j}^*) \cup \Phi^U(h_{i_j}^*, w_{i_j}^*)| \le 6N) \ge \varepsilon^4.$$

It is impossible and the reason is similar to that of (3.6).

Let $h^* = h_{i^*}^*$, $w^* = w_{i^*}^*$. Then by (3.9),

$$\mathbf{P}_\lambda^p(|\Phi^{UL}(h^*, w^*)| > N) = \mathbf{P}_\lambda^p(|\Phi^{UR}(h^*, w^*)| > N) > 1 - \varepsilon. \tag{3.10}$$

So (2) is true. Then we have proved the lemma. □

In order to state our result concisely, we introduce a special notation $\langle \cdot, \cdot \rangle$. For $a, b, c, d \in \mathbb{Z}$, define

$$\langle a + b\mathrm{i}, c + d\mathrm{i}\rangle := \begin{cases} \{(u, v) \in \mathbb{E} : u, v \in \lceil a + b\mathrm{i}, c + d\mathrm{i}\rfloor, \{\Re(u), \Re(v)\} \not\subseteq \{a, c\}\}, & \text{if } |a - c| \ge 2|b - d|,\\ \{(u, v) \in \mathbb{E} : u, v \in \lceil a + b\mathrm{i}, c + d\mathrm{i}\rfloor, \{\Im(u), \Im(v)\} \not\subseteq \{b, d\}\}, & \text{if } 2|a - c| \le |b - d|. \end{cases}$$

Then $\langle a + b\mathrm{i}, c + d\mathrm{i}\rangle$ is an edge set. See Fig. 6. Let E be the event that 0×0 is joined to every site of $\lceil -r + 4r\mathrm{i}, r + 4r\mathrm{i}\rfloor \times 1$ within $\langle -r, r + 4r\mathrm{i}\rangle$. Fix $N \ge \frac{20r \log \varepsilon}{\log(1 - \mathbf{P}_\lambda^p(E))} + 1$ which is large enough to ensure that in $[N/20r]$ or more independent trials of an experiment with success probability $\mathbf{P}_\lambda^p(E)$, the probability of obtaining at least one success exceeds $1 - \varepsilon$.

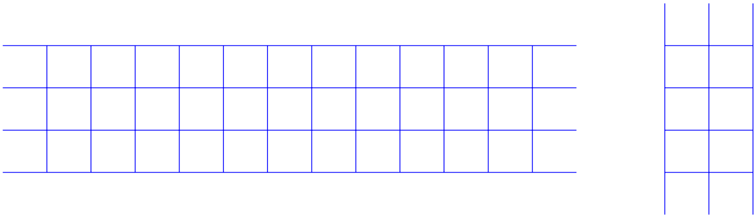


Fig. 6 $\langle a+b\mathrm{i}, c+d\mathrm{i}\rangle$

Lemma 3.3 *Suppose* $\mathbf{P}_\lambda^p(|\Phi^R(h,w)|>N)>1-\varepsilon$. *Then with* $\mathbf{P}_\lambda^p$*-probability greater than* $1-2\varepsilon$, *there exist* $x\in\lceil w+4r, w+4r+h\mathrm{i}\rfloor$ *and* $t>0$, *such that the horizontal seed* $(0\times 0)_r$ *is joined to the vertical seed* $(x\times t)_r$ *within* $\langle -w-1, w+4r+h\mathrm{i}\rangle$.

Proof Let t_1 be the first time that some site in $\lceil w+2r\mathrm{i}, w+(h-2r)\mathrm{i}\rfloor$ is infected. Precisely,

$$t_1 := \inf\{t : \lceil -r, r\rfloor \times 0 \text{ is joined to } \lceil w+2r\mathrm{i}, w+(h-2r)\mathrm{i}\rfloor \times t \text{ within } \lceil -w, w+h\mathrm{i}\rfloor\}.$$

If $t_1<\infty$, then with probability 1 there exists a unique infected site $x_1\in\lceil w+2r\mathrm{i}, w+(h-2r)\mathrm{i}\rfloor$ such that $\lceil -r, r\rfloor\times 0$ is joined to $x_1\times t_1$ within $\lceil -w, w+h\mathrm{i}\rfloor$. Generally, let t_k be the first time that some site in $\lceil w+2r\mathrm{i}, w+(h-2r)\mathrm{i}\rfloor\backslash(\bigcup_{i=1}^{k-1}\lceil x_i-3r\mathrm{i}, x_i+3r\mathrm{i}\rfloor)$ is infected, and x_k be the corresponding infected site if $t_k<\infty$. Denote by E_k the event that $x_k\times t_k$ is joined to every site of $\lceil x_k+4r-r\mathrm{i}, x_k+4r+r\mathrm{i}\rfloor\times(t_k+1)$ within $\langle x_k-r\mathrm{i}, x_k+4r+r\mathrm{i}\rangle$. If E_k occurs, then the horizontal seed $(0\times 0)_r$ is joined to the vertical seed $(x_k\times t_k)_r$ within $\langle -w-1, w+4r+h\mathrm{i}\rangle$. By the transitivity and the rotation invariance, we know that $(1_{E_k}|t_k<\infty)$ has the same distribution with 1_E. Let

$$Y_k = \begin{cases} 1_{E_k}, & \text{if } t_k<\infty, \\ \text{an independent random variable with the distribution } 1_E, & \text{if } t_k=\infty. \end{cases}$$

Then $\mathbf{P}(Y_k=1)=1-\mathbf{P}(Y_k=0)=\mathbf{P}_\lambda^p(E)$.

By the strong Markov property and the translation invariance, $Y_1, Y_2, \ldots$ are independent with respect to $\mathbf{P}_\lambda^p$. If $|\Phi^R(h,w)|>N$, then $t_1<\cdots<t_{[N/20r]}<\infty$ almost surely. Directly calculate

$$\begin{aligned}
\mathbf{P}_\lambda^p(\text{some } E_k \text{ occurs}) &\geq \mathbf{P}_\lambda^p\left(\sum_{k=1}^{[N/20r]} 1_{E_k}\geq 1\right) \\
&\geq \mathbf{P}_\lambda^p\left(|\Phi^R(h,w)|>N, \sum_{k=1}^{[N/20r]} Y_k\geq 1\right) \\
&\geq \mathbf{P}_\lambda^p(|\Phi^R(h,w)|>N)+\mathbf{P}_\lambda^p\left(\sum_{k=1}^{[N/20r]} Y_k\geq 1\right)-1 \\
&\geq 1-2\varepsilon.
\end{aligned}$$

So there exist $x\in\lceil w+4r+2r\mathrm{i}, w+4r+(h-2r)\mathrm{i}\rfloor$ and $t>0$, such that the horizontal seed $(0\times 0)_r$ is joined to the vertical seed $(x\times t)_r$ within $\langle -w-1, w+4r+h\mathrm{i}\rangle$ with $\mathbf{P}_\lambda^p$-probability greater than $1-2\varepsilon$. $\square$

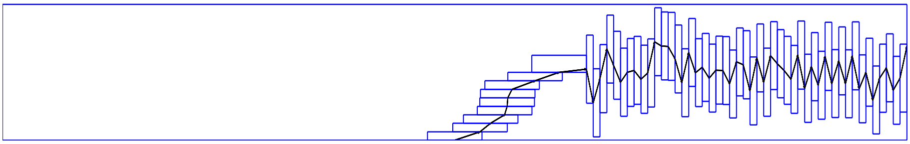


Fig. 7 Construction of (1) through (2)

Remark Similar conclusion holds for Φ^L, Φ^{UR} and Φ^{UL}.

Lemma 3.4 *Suppose* $p < 1$ *and* $\mathbf{P}_\lambda^p(\xi^0$ *survives*$) > 0$. *Then for any* $\varepsilon > 0$, *there exist* $r \geq 1$ *and* $h \geq 100r$ *such that the following three assertions hold with* $\mathbf{P}_\lambda^p$*-probability greater than* $1-\varepsilon$:

(i) *the horizontal seed* $(0\times 0)_r$ *is joined to a vertical seed* $(x \times t)_r$ *within* $\langle -4h-1, w+h\mathrm{i}\rangle$ *for some* $4h+4r \leq w < 4.0001h$, $\Re(x) = w$ *and* $t > 0$;
(ii) *the horizontal seed* $(0\times 0)_r$ *is joined to a vertical seed* $(x \times t)_r$ *within* $\langle -8h-1, w+h\mathrm{i}\rangle$ *for some* $8h+4r \leq w < 8.0001h$, $\Re(x) = w$ *and* $t > 0$;
(iii) *the horizontal seed* $(0 \times 0)_r$ *is joined to a vertical seed* $(x_1 \times t_1)_r$ *within* $\langle -8h-1, w_1+h\mathrm{i}\rangle$ *for some* $8h+4r \leq w_1 < 8.0001h$ *and* $t_1 > 0$; *and the horizontal seed* $(0 \times 0)_r$ *is joined to a vertical seed* $(x_2 \times t_2)_r$ *within* $\langle -w_2+h\mathrm{i}, 8h+1\rangle$ *for some* $8h+4r \leq w_2 < 8.0001h$ *and* $t_2 > 0$.

Proof When $p < 1$ and $\mathbf{P}_\lambda^p(\xi^0$ survives$) > 0$, either (1) or (2) of Lemma 3.2 is true. If (1) is true, then by Lemma 3.3, (i) and (ii) hold. If (2) is true, we can prove the first two conclusions by iterating Lemma 3.3 and the remark below Lemma 3.3, see Fig. 7. Furthermore, by (ii), the symmetric property and the FKG inequality, we can get (iii) in both cases. So we have completed the proof of the lemma. □

We are going to introduce random variables S_T, S_S, L_T and L_S, together with *S-boxes* and *L-boxes*.

For $x, y \in \mathbb{H}$, $t > s > 0$ and $w \in \lceil 4h+4r, [4.0001h]\rfloor$, let

$$(x \times s)_r \xrightarrow{1,1,w} (y \times t)_r$$

be the event that $\Re(y-x) = w$ and $\lceil x-r, x+r\rfloor \times s$ is joined within $\langle x-4h-1, x+w+h\rangle$ to $z \times t$ for all $z \in \lceil y-r\mathrm{i}, y+r\mathrm{i}\rfloor$. Define random time

$$S_T(x, s, 1, 1) = \inf\{t : \exists y,\ (x \times s)_r \xrightarrow{1,1,w^*} (y \times t)_r\},$$

where

$$w^* = \inf\{w \in [4h+4r, 4.0001h) : \exists z\ \exists u,\ (x \times s)_r \xrightarrow{1,1,w} (z \times u)_r\}.$$

If such w^* does not exist, then set $S_T(x, s, 1, 1) = \infty$. If $S_T(x, s, 1, 1) < \infty$, define $S_S(x, s, 1, 1)$ taken value from $\{y : (x \times s)_r \xrightarrow{1,1,w^*} (y \times S_T(x, s, 1, 1))_r\}$ in a certain way. As a result, if $S_T(x, s, 1, 1) < \infty$, then $(x \times s)_r$ is joined to $(S_S(x, s, 1, 1) \times S_T(x, s, 1, 1))_r$ within $\langle x-4h-1, x+w^*+h\mathrm{i}\rangle$, where $w^* = \Re(S_S(x, s, 1, 1)) \in \lceil 4h+4r, [4.0001h]\rfloor$. We call such $\langle x-4h-1, x+w^*+h\mathrm{i}\rangle$ an *S-box*.

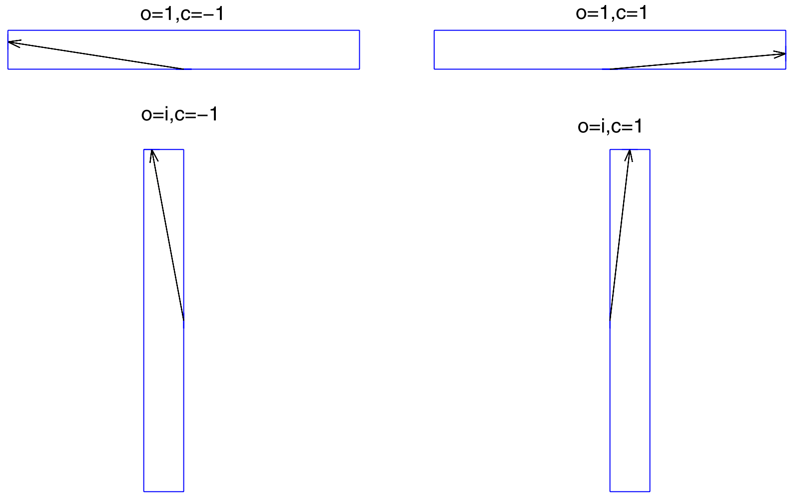


Fig. 8 Orientations in S-boxes and L-boxes

Similarly define $L_T(x,t,1,1)$ and $L_S(x,t,1,1)$, so that if $L_T(x,s,1,1)<\infty$, then $(x\times s)_r$ is joined within $\langle x-8h-1, x+w^*+h\mathrm{i}\rangle$ to $(L_S(x,s,1,1)\times L_T(x,s,1,1))_r$, where $w^*=\Re(L_S(x,s,1,1))\in\lceil 8h+4r,[8.0001h]\rfloor$. We call such $\langle x-8h-1, x+w^*+h\mathrm{i}\rangle$ an *L-box*.

Similarly define $S_S(x,t,o,c)$, $S_T(x,t,o,c)$, $L_S(x,t,o,c)$, $L_T(x,t,o,c)$ for $o\in\{1,\mathrm{i}\}$, $c\in\{1,-1\}$, see Fig. 8.

Lemmas 3.5 and 3.6 below show that S_S, S_T, L_S and L_T can help us find restart processes which are independent of the former process in the annealed law. They are crucial to Proposition 4.1.

For $A\subseteq\mathbb{H}$, $\mathbb{G}\subseteq\mathbb{E}$ and $t>s$, define $(\xi_t^{A,s,\mathbb{G}}, t\ge s)$ on $(\Omega^s,\mathcal{F},\mathbf{P}_\lambda)$ by

$$\xi_t^{A,s,\mathbb{G}}=\xi_t^{A,s,\mathbb{G}}(\mathcal{C})=\{x: A\times s \text{ is joined within } \mathbb{G} \text{ to } x\times t\},$$

which is the contact process restricted on $\mathbb{G}$ starting at time s with initial state A. For $y\in\mathbb{H}$, define

$$y+\mathbb{G}:=\{(u+y,v+y):(u,v)\in\mathbb{G}\},$$

a shift of $\mathbb{G}$. For $y\in\mathbb{H}$ and $u\ge 0$, define process

$$\tilde{\xi}_t^{y,\,\mathbb{G}}\circ\theta_u:=\xi_{t+u}^{\lceil y-r\mathrm{i},y+r\mathrm{i}\rfloor,u,y+\mathbb{G}}-y,\qquad \forall t\ge 0.$$

Then $\tilde{\xi}^{y,\,\mathbb{G}}\circ\theta_u+y$ is the contact process restricted on $(y+\mathbb{G})$ starting from time u with initial state $\lceil y-r\mathrm{i}, y+r\mathrm{i}\rfloor$. Write $\tilde{\xi}_t^{y,\,\mathbb{G}}=\tilde{\xi}_t^{y,\,\mathbb{G}}\circ\theta_0$ for simplicity.

Lemma 3.5 *Let $\mathbb{G}$ be an edge set and $x\in\mathbb{H}$. Suppose $(x+w+a\mathrm{i})+\mathbb{G}\subseteq\mathbb{E}$ shares no edges with $\langle x-4h-1, x+w+h\mathrm{i}\rangle$ for all $w\in\lceil 4h+4r,[4.0001h]\rfloor$ and $a\in\lceil 0,h\mathrm{i}\rfloor$. Then*

$$\mathbf{P}_\lambda^p(S\in A, T\in B,\ \tilde{\xi}^{S,\mathbb{G}}\circ\theta_T\in C\ |T<\infty)=\mathbf{P}_\lambda^p(S\in A, T\in B|T<\infty)\mathbf{P}_\lambda^p(\tilde{\xi}^{x,\mathbb{G}}\in C)$$

provided the left side is meaningful, where $S=S_S(x,0,1,1)$, $T=S_T(x,0,1,1)$.

Lemma 3.6 *Let $\mathbb{G}_{-1}$ and $\mathbb{G}_1$ be two edge sets and $x \in \mathbb{H}$. Suppose $(x + w_i + a_i\mathsf{i}) + \mathbb{G}_i \subseteq \mathbb{E}$ shares no edges with $\langle x - 8ih - i, x + iw_i + h\mathsf{i}\rangle$ and no endpoints with $(-iw_{-i} + a_{-i}\mathsf{i} + G_{-i}) \cup \langle x + 8ih + i, x - iw_{-i} + h\mathsf{i}\rangle$ for $i \in \{1, -1\}$, $w_1, w_{-1} \in \lceil 8h + 4r, [8.0001h]\rfloor$ and $a_1, a_{-1} \in \lceil 0, h\mathsf{i}\rfloor$. Let $S_1 = S_S(x, 0, 1, 1)$, $S_{-1} = S_S(x, 0, 1, -1)$, $T_1 = S_T(x, 0, 1, 1)$ and $T_{-1} = S_T(x, 0, 1, -1)$. Then*

$$\begin{aligned}
&\mathbf{P}^p_\lambda(S_1 \in A_1,\ S_{-1} \in A_{-1},\ T_1 \in B_1,\ T_{-1} \in B_{-1},\ \tilde{\xi}^{S_1,\mathbb{G}_1} \circ \theta_{T_1} \in C_1,\\
&\quad \tilde{\xi}^{S_{-1},\mathbb{G}_{-1}} \circ \theta_{T_{-1}} \in C_{-1} | T_1 < \infty, T_{-1} < \infty)\\
&\quad = \mathbf{P}^p_\lambda(S_1 \in A_1,\ S_{-1} \in A_{-1},\ T_1 \in B_1,\ T_{-1} \in B_{-1} | T_1 < \infty, T_{-1} < \infty)\mathbf{P}^p_\lambda(\tilde{\xi}^{x,\ \mathbb{G}_1} \in C_1)\\
&\quad\quad \times \mathbf{P}^p_\lambda(\tilde{\xi}^{x,\ \mathbb{G}_{-1}} \in C_{-1})
\end{aligned}$$

provided the left side is meaningful.

The proof of Lemmas 3.5 and 3.6 will be deferred to Appendix 1.

4 Construction of Integrated Boxes

We shall define a number of mappings in Sects. 4 and 5. Every mapping corresponds to an algorithm. More precisely,

$$\mathring{A}_k : \vec{x} \mapsto \mathring{A}_k(\vec{x}),$$

where $\vec{x}$ is the input and $\mathring{A}_k(\vec{x})$ is the output by Algorithm k. For each $\varepsilon > 0$, fix $r = r(\varepsilon)$ and $h = h(\varepsilon)$ satisfying Lemma 3.4 henceforth. We always write $\mathrm{M} = 10^7$ in this paper. For $x \in \mathbb{H}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, define

$$R_{m,n} := \lceil a + m\mathrm{M}h + n\mathrm{M}h\mathsf{i}, b + m\mathrm{M}h + n\mathrm{M}h\mathsf{i}\rfloor = \lceil a, b\rfloor + \mathrm{M}h(m + n\mathsf{i}),$$

where $a = 100h[\Re(x)/100h] + 100h[\Im(x)/100h]\mathsf{i}$ and $b = a + 100(1 + \mathsf{i})$. Then $R_{m,n}$ is a square and $x \in R_{0,0}$.

Suppose $(x \times s)_r$ is a seed. We first introduce Algorithm 1, which is used to construct a route by which the seed $(x \times s)_r$ is joined to other seeds in $R_{0,1}$ and $R_{1,0}$ with large probability, see Figs. 2 and 3 respectively.

Algorithm 1

0) Set $i = 1$, $(y, t) = (x, s)$ and $t_1 = t_2 = \infty$.
1) If $S_T(y, t, \mathsf{i}, -1) < \infty$, then set $(y, t) = (S_S, S_T)(y, t, \mathsf{i}, -1)$ and $i = i + 1$. Otherwise go to 17).
2) If $S_T(y, t, 1, -1) < \infty$, then set $(y, t) = (S_S, S_T)(x, s, 1, -1)$ and $i = i + 1$. Otherwise go to 17).
3) If $y \in \lceil a + \mathrm{M}h\mathsf{i} + 50h, a + \mathrm{M}h\mathsf{i} + 60h + 90h\mathsf{i}\rfloor$, then go to 14).
4) If $\Re(y) \geq \Re(a) + 30h$, then go to 1).
5) If $L_T(y, t, \mathsf{i}, -1) < \infty$, then set $(y, t) = (L_S, L_T)(y, t, \mathsf{i}, -1)$ and $i = i + 1$. Otherwise go to 17).
6) If $L_T(y, t, 1, 1) < \infty$, then set $(y, t) = (L_S, L_T)(y, t, 1, 1)$ and $i = i + 1$. Otherwise go to 17).

7) If $S_T(y,t,\mathrm{i},1)<\infty$, then set $(y,t)=(S_S,S_T)(y,t,\mathrm{i},1)$ and $i=i+1$. Otherwise go to 17).
8) If $S_T(y,t,1,1)<\infty$, then set $(y,t)=(S_S,S_T)(y,t,1,1)$ and $i=i+1$. Otherwise go to 17).
9) If $y\in\lceil a+\mathrm{M}h\mathrm{i}+40h, a+\mathrm{M}h\mathrm{i}+50h+90h\mathrm{i}\rfloor$, then go to 15).
10) If $\Re(y)\le\Re(a)+70h$, then go to 7).
11) If $L_T(y,t,\mathrm{i},1)<\infty$, then set $(y,t)=(L_S,L_T)(y,t,\mathrm{i},1)$ and $i=i+1$. Otherwise go to 17).
12) If $L_T(y,t,1,-1)<\infty$ then set $(y,t)=(L_S,L_T)(y,t,1,-1)$ and $i=i+1$. Otherwise go to 17).
13) Go to 1).
14) If $L_T(y,t,\mathrm{i},-1)<\infty$, set $(y,t)=(L_S,L_T)(y,t,\mathrm{i},-1)$, $i=i+1$, and go to 16). Otherwise go to 17).
15) If $L_T(y,t,\mathrm{i},1)<\infty$, then set $(y,t)=(L_S,L_T)(y,t,\mathrm{i},1)$ and $i=i+1$. Otherwise go to 17).
16) If $L_T(y,t,1,-1)+L_T(y,t,1,1)<\infty$, set $(y_1,t_1)=(L_S,L_T)(y,t,1,-1)$ and $(y_2,t_2)=(L_S,L_T)(y,t,1,1)$.
17) Return t_1,t_2,y_1,y_2,i.

Here, $(f,g)(y,t,o,c)=(f(y,t,o,c),g(y,t,o,c))$. The idea of Algorithm 1 is as follows. Use S-boxes (horizontal and vertical boxes alternatively) to let the seed spread in the northwest ('$\nwarrow$') direction. If the infection surpasses the line $\{y:\ \Re(y)=\Re(a)+30h\}$, then use two L-boxes to change the spread into the northeast ('$\nearrow$') direction. If the infection surpasses the line $\{y:\ \Re(y)=\Re(a)+70h\}$, then use two L-boxes to change the spread into the northwest direction. Iterate the procedure until $R_{0,1}$ is infected. Then use two L-boxes to get the two infected seeds we want. As a result, by the route given by Algorithm 1, the vertical seed $(x\times s)_r$ may be joined to two vertical seeds $(y_1\times t_1)_r$ and $(y_2\times t_2)_r$, where $y_1,y_2\in R_{0,1}$. Note that the route lies in $\lceil a,b+\mathrm{M}h\rfloor$.

Algorithm 1 must end with $i<\mathrm{M}$. So $t_1+t_2<\infty$ with large probability by Lemmas 3.4, 3.5 and 3.6. If $t_1+t_2<\infty$, it generates two seeds as our requirement. Define mapping

$$\mathring{A}_1^v(s,x,\mathrm{i}):=(t_1,t_2,y_1,y_2).$$

Here '1' corresponds to Algorithm 1, 'v' corresponds to that the seed at the initial state is vertical, and 'i' corresponds to that the orientation of the infection is north ('$\uparrow$'). In application we care little about the precise values of x,y_1 and y_2. Moreover, y_1 and y_2 are 'almost' determined by t_1 and t_2 respectively. Hence we omit the space parameters and abuse the notation $\mathring{A}_1^v(s,\mathrm{i})=(t_1,t_2)$. Similarly, we shall omit the space parameters for $\mathring{A}_i$ $(i=2,\dots,6)$, which will be defined later.

Define mapping $\mathring{A}_1^v(s,1):=(t_1',t_2')$ similarly, see Fig. 3. A little difference is that $\mathring{A}_1^v(s_0,1)$ generates two horizontal seeds $(y_1'\times t_1')_r,(y_2'\times t_2')_r$ with $y_1,y_2\in R_{1,0}$ and $\Im(y_1'-y_2')>0$. Define $\mathring{A}_1^h$ for the case that the seed at the initial state is horizontal in the same way. In application we care little about whether the seed at the initial state is vertical or horizontal, so we simply write $\mathring{A}_1$.

Next we introduce Algorithm 2. By the algorithm we construct integrated boxes to get a route, as shown in Fig. 4. We may get some $y,z\in R_{n,n}$ and $t,u<\infty$ by the route, such that the seed $(x\times s)_r$ is joined to the seeds $(y\times t)_r$ and $(z\times u)_r$ within $\lceil a,b+n\mathrm{M}h(1+\mathrm{i})\rfloor$.

Algorithm 2

1) Set $(t_{0,1}^{\mathrm{i}}, t_{0,1}^{1}) = \mathring{A}_1(s, x, \mathrm{i})$;
2) For $2 \le j \le n$
   set $(t_{0,j}^{\mathrm{i}}, t_{0,j}^{1}) = \mathring{A}_1(t_{0,j-1}^{\mathrm{i}}, \mathrm{i})$;
   End
3) Set $t_{j,0}^{\mathrm{i}} = \infty$, $j = 1, \dots, n$;
4) For $1 \le i \le n$
   For $1 \le j \le n$
   set $(t_{i,j}^{\mathrm{i}}, t_{i,j}^{1}) = \mathring{A}_1(t_{i-1,j}^{1}, 1)1_{\{t_{i-1,j}^{1} < \infty\}} + \mathring{A}_1(t_{i,j-1}^{\mathrm{i}}, \mathrm{i})1_{\{t_{i-1,j}^{1} = \infty\}}$;
   End
   End
5) Return $t_{n,n}^{\mathrm{i}}, t_{n,n}^{1}$.

If $t_{n,n}^{\mathrm{i}} + t_{n,n}^{1} < \infty$, then there exist $y, z \in R_{n,n}$, such that the seed $(x \times s)_r$ is joined to two seeds $(y \times t_{n,n}^{\mathrm{i}})_r$ and $(z \times t_{n,n}^{1})_r$ within $\lceil a, b + n\mathrm{M}h(1 + \mathrm{i})\rfloor$. Define

$$\mathring{A}_L(s, x, n, 1 + \mathrm{i}) := t_{n,n}^{\mathrm{i}} \quad \text{and} \quad \mathring{A}_R(s, x, n, 1 + \mathrm{i}) := t_{n,n}^{1},$$

where $1 + \mathrm{i}$ indicates that the orientation of infection is northeast.

Similarly we can define $\mathring{A}_L(s, x, n, o) = t_1$, $\mathring{A}_R(s, x, n, o) = t_2$ for $o \in \{1 - \mathrm{i}, -1 + \mathrm{i}, -1 - \mathrm{i}\}$. If $t_1 + t_2 < \infty$, then there exist $x_1, x_2 \in \lceil a, b\rfloor + n\mathrm{M}ho$, such that the seed $(x \times s)_r$ is joined to two seeds $(x_1 \times t_1)_r$ and $(x_2 \times t_2)_r$, and x, x_1, x_2 are arranged *clockwise*.

Having introduced Algorithms 1 and 2, we can state the main proposition in this section now.

Proposition 4.1 *Suppose $p < 1$ and $\mathbf{P}_\lambda^p(\xi^0$ survives$) > 0$. Let $x = x(\varepsilon) \in \mathbb{H}$ with $\Im(x) > 10h$ and $(x \times 0)_r$ be a horizontal seed. Then there exists $\overline{W} > 0$ which depends only on ε, p and λ, such that*

$$\lim_{\varepsilon \to 0+} \liminf_{n \to \infty} \mathbf{P}_\lambda^p\left(\frac{7\overline{W}}{6}n < \mathring{A}_L(0, x, n, 1 + \mathrm{i}) < \frac{11\overline{W}}{6}n\right) = 1$$

and

$$\lim_{\varepsilon \to 0+} \liminf_{n \to \infty} \mathbf{P}_\lambda^p\left(\frac{7\overline{W}}{6}n < \mathring{A}_R(0, x, n, 1 + \mathrm{i}) < \frac{11\overline{W}}{6}n\right) = 1.$$

Proposition 4.1 will be proved in Appendix 2. We only state the idea here. Algorithm 1 provides a route by which a seed in $R_{m,n}$ is joined to other seeds in $R_{m+1,n}$ and $R_{m,n+1}$ with large probability. As a result, we use the renormalization method and consider each $R_{m,n}$ as one site. Declare $R_{0,0}$ open if $x \in R_{0,0}$ and $(x \times 0)_r$ is a seed. For $m + n \ge 1$, declare $R_{m,n}$ open if and only if

(i) $R_{m-1,n}$ is open and the seed in $R_{m-1,n}$ is joined to two seeds in $R_{m,n}$ through Algorithm 1, or
(ii) $R_{m-1,n}$ is closed, $R_{m,n-1}$ is open and the seed in $R_{m,n-1}$ is joined to two seeds in $R_{m,n}$ through Algorithm 1.

Refer to Fig. 4. The process $(R_{m,n})$ is an oriented site percolation. Refer to Grimmett [6]. We can find a unique open path from $R_{0,0}$ to $R_{n,n}$ with large probability. Furthermore, we can find the unique route constructed by S-boxes and L-boxes, within which the seed in $R_{0,0}$

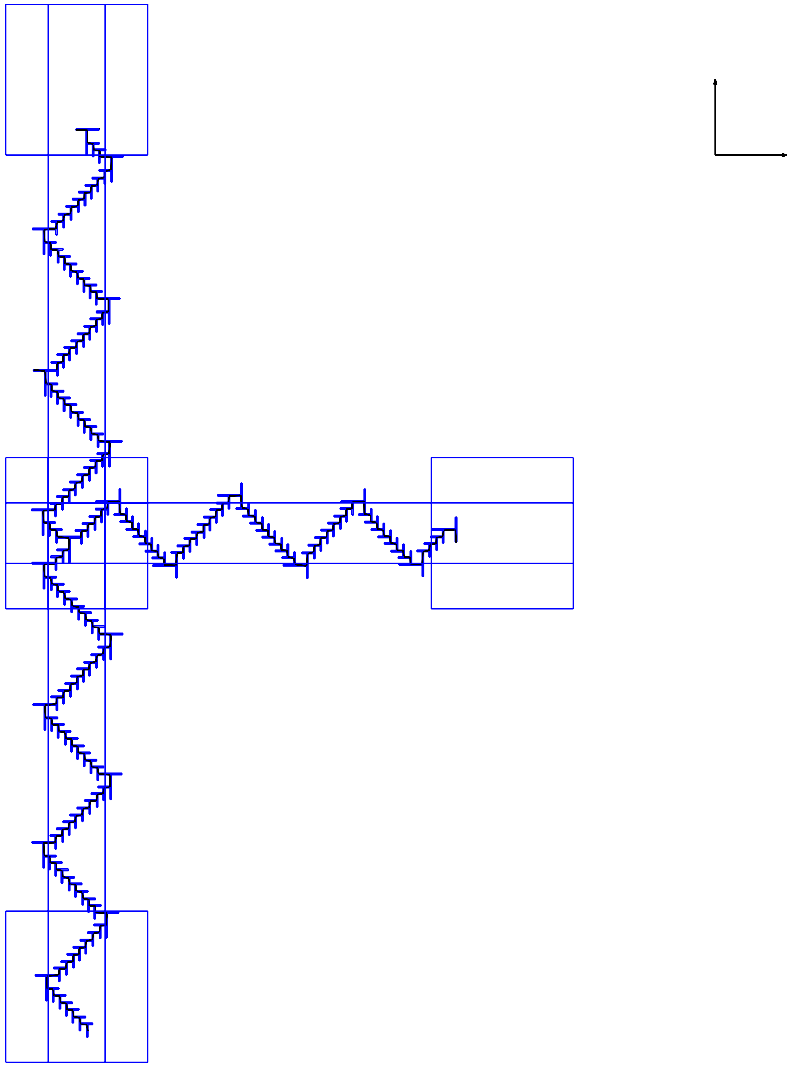


Fig. 9 All S-boxes are disjoint

is joined to another two seeds in $R_{n,n}$. It implies that $\mathring{A}_L(\mathring{A}_R)$ is the sum of the time spent in each box. Figure 9, which describes the way how to get $\mathring{A}_1(s, x, \mathrm{i})$, $\mathring{A}_1(t^{\mathrm{i}}_{0,1}, \mathrm{i})$ and $\mathring{A}_1(t^1_{0,1}, 1)$, shows that all S-boxes used in Algorithm 2 are disjoint. So the times spent in each box are independent under certain condition by Lemmas 3.5 and 3.6. Through rigorous calculation, we get that the total number of S-boxes on the route is between $2nj_{lower}$ and $2nj_{upper}$. Then by the law of large numbers, the time spent in these S-boxes is almost between $\frac{7}{6}\overline{S}n$ and $\frac{11}{6}\overline{S}n$. We can deduce that the time spent in these L-boxes is almost between $\frac{7}{6}\overline{L}n$ and $\frac{11}{6}\overline{L}n$, too. Hence the total time $\mathring{A}_L(\mathring{A}_R)$ is almost between $\frac{7}{6}\overline{W}n$ and $\frac{11}{6}\overline{W}n$. Here j_{lower} and j_{upper} are two constants which satisfy $1 \le j_{upper}/j_{lower} < \frac{11}{6}$, and $\overline{S}, \overline{L}, \overline{W}$ depend only on p, λ and ε.

5 The Complete Convergence Theorem

In this section we shall prove the complete convergence theorem. When $p=1$ the theorem is already proved in Bezuidenhout & Grimmett [1]. Therefore, throughout this section we suppose $p<1$ and $\mathbf{P}_\lambda^p(\xi^0 \text{ survives})>0$. It is easy to prove that $\xi_t^{A\setminus \mathcal{C}_\infty}$ converges weakly to $\delta_\emptyset$ for all $A\subset \mathbb{H}$ and almost all ω. Therefore, ξ^A and $\xi^{A\cap\mathcal{C}_\infty}$ have the same limit behavior. Furthermore, by Theorem 1.12 of Liggett [10], to prove Theorem 1.1 it suffices to prove that there exists $\Omega_0\subseteq \Omega^b$ with $\mathbf{P}^p(\Omega_0)=1$ such that for all $\omega\in\Omega_0$,

(i) $\mathbf{P}_\lambda(x\in \limsup \xi_t^A(\mathcal{C}))=\mathbf{P}_\lambda(\xi^A(\mathcal{C}) \text{ survives})$ for all $x\in\mathcal{C}_\infty$ and $A\subset\mathcal{C}_\infty$;

(ii) $\lim_{l\to\infty}\liminf_{t\to\infty}\mathbf{P}_\lambda(\xi_t^{B_0(l)}(\mathcal{C})\cap B_0(l)\neq\emptyset)=1$.

We shall prove (i) first. Fix A a nonempty finite subset of $\mathbb{H}$. Let x_0 be any element of A and $\sigma_0=0$. Hence x_0 is infected at time σ_0 during the process ξ^A. Define $\delta_k,\tau_k,Y_k,\sigma_{k+1}$ and x_{k+1} inductively for $k\geq 0$ as follows. Let

$$\begin{aligned}\delta_k := \sup\{t : x_k\times\sigma_k \text{ is joined within } &\langle x_k-r-1, x_k+r+1+2000h\mathrm{i}\rangle\\ &\text{to } \lceil x_k-r-1, x_k+r+1+2000h\mathrm{i}\rfloor\times t\}\end{aligned}$$

be the death time of $\xi^{\{x_k\},\ \sigma_k,\langle x_k-r-1,x_k+r+1+2000h\mathrm{i}\rangle}$. Then $\delta_k<\infty$ almost surely. Let

$$\begin{aligned}\tau_k := \min\{t-\sigma_k : x_k\times\sigma_k \text{ is joined within } &\langle x_k-r-1, x_k+r+1+2000h\mathrm{i}\rangle\\ &\text{to } z\times t \text{ for all } z\in\lceil x_k-r+2000h\mathrm{i}, x_k+r+2000h\mathrm{i}\rfloor\}\end{aligned}$$

be the waiting time for the first seed on the top. Then $\mathbf{P}_\lambda^p(\tau_k<\infty|\sigma_k<\infty)>0$, and $\sigma_k+\tau_k<\delta_k$ if $\tau_k<\infty$. Let

$$Y_k := \sup\left\{\Im(x)+2 : x\in\bigcup_{t\leq\delta_k}\xi_t^A\right\}.$$

Hence $Y_k<\infty$ almost surely. Let

$$\sigma_{k+1} := \inf\{t>\delta_k : \exists x\in\xi_t^A,\ \Im(x)=Y_k\}$$

and x_{k+1} be the corresponding infected site.

$\{\tau_k : k\geq 0\}$ are independent and identically distributed random variables conditioned on $\sigma_k<\infty$ for all k. Hence $K:=\min\{k:\tau_k<\infty\}<\infty$ almost surely conditioned on $\sigma_k<\infty$ for all k. But ξ^A survives almost surely if and only if $\sigma_k<\infty$ for all k, which means that

$$\mathbf{P}_\lambda^p(K<\infty|\ \xi^A \text{ survives})=1. \tag{5.1}$$

Therefore, $(x_K\times\tau_K)_r$ is a horizontal seed. Let

$$\zeta=\mathring{A}_L(\mathring{A}_L(\mathring{A}_L(\mathring{A}_L(\tau_K,x_K,m,1+\mathrm{i}),m,-1+\mathrm{i}),m,-1-\mathrm{i}),m-1,1-\mathrm{i}),$$

and $(\vartheta\times\zeta)_r$ be the corresponding seed if $\zeta<\infty$. Then

$$\vartheta\in R_{-1,1}\subset B_{x_K}(2\mathrm{M}h).$$

Refer to Fig. 5. By Proposition 4.1 and the strong Markov property,

$$\lim_{\varepsilon\to 0+}\liminf_{m\to\infty}\mathbf{P}_\lambda^p(3\overline{W}m\leq\zeta<\infty|1_{\{K<\infty\}},x_K)=1,$$

which implies that

$$\lim_{\varepsilon\to 0+}\liminf_{m\to\infty}\mathbf{P}_\lambda^p(\exists t\geq 3\overline{W}m, \xi_t^A\cap B_{x_K}(2\mathrm{M}h)\neq\emptyset|1_{\{K<\infty\}}, x_K)=1.$$

By the dominated convergence theorem,

$$\lim_{\varepsilon\to 0+}\mathbf{P}_\lambda^p\Big(\limsup_{t\to\infty}\xi_t^A\cap B_{x_K}(2\mathrm{M}h)\neq\emptyset|1_{\{K<\infty\}}, x_K\Big)=1.$$

Furthermore,

$$\lim_{\varepsilon\to 0+}\mathbf{P}_\lambda^p\Big(\limsup_{t\to\infty}\xi_t^A\neq\emptyset|1_{\{K<\infty\}}\Big)=1. \tag{5.2}$$

By (5.1) and (5.2),

$$\mathbf{P}_\lambda^p\Big(\limsup_{t\to\infty}\xi_t^A\neq\emptyset|\ \xi^A \text{ survives}\Big)=1.$$

Therefore, there exists $\Omega_A\subseteq\Omega^b$ with $\mathbf{P}^p(\Omega_A)=1$, such that for all $\omega\in\Omega_A$,

$$\mathbf{P}_\lambda\Big(\xi^A(\mathcal{C}) \text{ survives, and } \limsup_{t\to\infty}\xi_t^A(\mathcal{C})=\emptyset\Big)=0. \tag{5.3}$$

That is to say, $\xi^A(\mathcal{C})$ survives strongly if it survives. See p. 42 of Liggett [10] for the definition of strong survival.

Fix $\omega\in\Omega_A$. Suppose $\limsup_{t\to\infty}\xi_t^A(\mathcal{C})\neq\emptyset$ and $y\in\limsup_{t\to\infty}\xi_t^A(\mathcal{C})$. Then $y\in\mathcal{C}_\infty$ since contact process must die out on a finite set and $\mathcal{C}_\infty$ is the unique infinite open cluster of $\mathcal{C}$. If $z\in\mathcal{C}_\infty$, then there exists at least one open path from y to z in $\mathcal{C}_\infty$. It implies that

$$\mathbf{P}_\lambda(z\in\xi_1^y(\mathcal{C}))>0.$$

Since $y\in\limsup_{t\to\infty}\xi_t^A(\mathcal{C})$, we know by the strong Markov property that

$$\mathbf{P}_\lambda\Big(z\in\limsup_{t\to\infty}\xi_t^A(\mathcal{C})|y\in\limsup_{t\to\infty}\xi_t^A(\mathcal{C})\Big)=1.$$

By the arbitrariness of y,

$$\mathbf{P}_\lambda\Big(z\in\limsup_{t\to\infty}\xi_t^A(\mathcal{C})|\limsup_{t\to\infty}\xi_t^A(\mathcal{C})\neq\emptyset\Big)=1. \tag{5.4}$$

Together with (5.3) we can deduce that for a finite subset $A\subseteq\mathbb{H}$, $\mathcal{C}\in\Omega_A$ and $z\in\mathcal{C}_\infty$,

$$\mathbf{P}_\lambda\Big(z\in\limsup_{t\to\infty}\xi_t^A(\mathcal{C})|\ \xi^A(\mathcal{C}) \text{ survives}\Big)=1.$$

Now let $\Omega_0'=\bigcap_n\bigcap_{A\subset\mathbb{H},\ |A|=n}\Omega_A$. Then $\mathbf{P}^p(\Omega_0')=1$. Moreover, (i) holds for all $\omega\in\Omega_0'$, $A\subset\mathcal{C}_\infty$ with $|A|<\infty$.

Consider $|A|=\infty$. We can get that for $n>0$, there exists m_n such that for any $B\subset\mathbb{H}$ with $|B|\geq m_n$, $\mathbf{P}_\lambda^p(\xi^B$ survives$)>1-4^{-n}$ by a reason similar to the proof Lemma 3.1. It implies that

$$\mathbf{P}^p(\{\omega:\mathbf{P}_\lambda(\xi^B(\mathcal{C}) \text{ survives})\geq 1-2^{-n}\})\geq 1-2^{-n}.$$

Let $\Xi'_n := \{\omega : \mathbf{P}_\lambda(\xi^B(\mathcal{C}) \text{ survives}) \geq 1 - 2^{-n}\}$ and $\Omega''_0 = \Omega'_0 \cap \liminf_{n\to\infty} \Xi'_n$. Then $\mathbf{P}^p(\Omega''_0) = 1$. If $\omega \in \Omega''_0$, $A \subset \mathcal{C}_\infty$ and $|A| = \infty$, then let (A_n) be an increasing sequence of finite sets which satisfy $\lim_n A_n = A$ and $|A_n| > m_n$. Then for $x \in \mathcal{C}_\infty$,

$$\begin{aligned}\mathbf{P}_\lambda\Big(x \in \limsup_{t\to\infty} \xi_t^A(\mathcal{C})\Big) &\geq \lim_{n\to\infty} \mathbf{P}_\lambda\Big(x \in \limsup_{t\to\infty} \xi_t^{A_n}(\mathcal{C})\Big)\\ &= \lim_{n\to\infty} \mathbf{P}_\lambda\big(\xi_t^{A_n}(\mathcal{C}) \text{ survives}\big)\\ &\geq \lim_{n\to\infty} (1 - 2^{-n}) \to 1.\end{aligned}$$

But $\xi^A(\mathcal{C})$ survives with $\mathbf{P}_\lambda$-probability one. So $\mathbf{P}_\lambda(x \in \limsup \xi_t^A(\mathcal{C})) = \mathbf{P}_\lambda(\xi_t^A(\mathcal{C})$ survives$) = 1$.

We have completed the proof of (i). We will next prove (ii). Before that we will define two mappings

$$\begin{aligned}&\mathring{A}_3 : \overline{\mathbf{R}^+} \times \mathbb{H} \times \mathbb{Z}^+ \times \{1, \mathrm{i}\} \to \mathbf{R}^+ \quad \text{and}\\ &\mathring{A}_4 : \overline{\mathbf{R}^+} \times \mathbb{H} \times \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{1+\mathrm{i}, 1-\mathrm{i}, -1+\mathrm{i}, -1-\mathrm{i}\} \to \mathbf{R}^+\end{aligned}$$

through Algorithms 3 and 4 respectively. The behavior of $\mathring{A}_4$ is similar to $\mathring{A}_2$, and we do not state too much here.

Suppose $(x \times s)_r$ is a seed again.

Algorithm 3

0) Set $t = s$ and $y = x$.
1) Set $s' = s - 100\overline{W}n[s/100\overline{W}n]$, $v = 8 \cdot 1_{\{s' \leq 37\overline{W}n\}}$ and $u = 9 - v$. One can check that

$$s' + (6u + 10v) \cdot \left[\frac{7}{6}\overline{W}n, \frac{11}{6}\overline{W}n\right] \subseteq [100\overline{W}n, 200\overline{W}n).$$

Operate 2)$\sim$7) u times
2) $t = \mathring{A}_R(t, n, 1+\mathrm{i})$;
3) $t = \mathring{A}_L(t, n, 1-\mathrm{i})$;
4) $t = \mathring{A}_L(t, n, 1+\mathrm{i})$;
5) $t = \mathring{A}_L(t, n, -1+\mathrm{i})$;
6) $t = \mathring{A}_R(t, n-1, -1-\mathrm{i})$;
7) $t = \mathring{A}_R(t, n+1, -1+\mathrm{i})$; Operate 8)$\sim$17) v times
8) $t = \mathring{A}_R(t, n, 1+\mathrm{i})$;
9) $t = \mathring{A}_L(t, n, 1-\mathrm{i})$;
10) $t = \mathring{A}_R(t, n, 1+\mathrm{i})$;
11) $t = \mathring{A}_L(t, n, 1-\mathrm{i})$;
12) $t = \mathring{A}_L(t, n, 1+\mathrm{i})$;
13) $t = \mathring{A}_L(t, n, -1+\mathrm{i})$;
14) $t = \mathring{A}_R(t, n-1, -1-\mathrm{i})$;
15) $t = \mathring{A}_L(t, n, -1+\mathrm{i})$;
16) $t = \mathring{A}_R(t, n, -1-\mathrm{i})$;
17) $t = \mathring{A}_R(t, n+1, -1+\mathrm{i})$;
18) Return t.

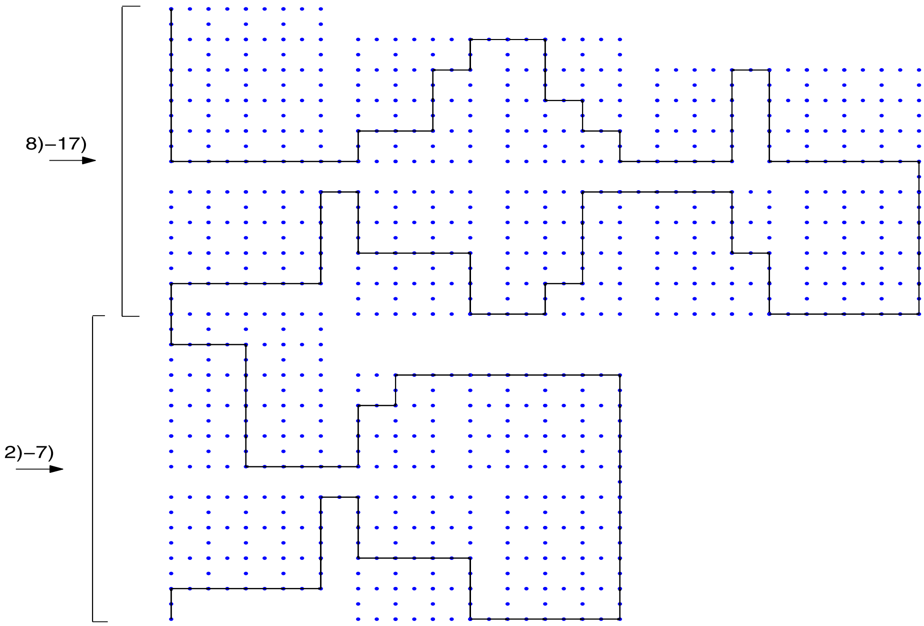


Fig. 10 $\mathring{A}_3(s, x, n, \mathsf{i})$

Refer to Fig. 10 for intuition. If $t < \infty$, then the corresponding site belongs to $R_{18(n+1),0}$. Moreover, by Proposition 4.1 we know that $t \in [100\overline{W}n, 200\overline{W}n)$ with large probability if $s \in [0, 100\overline{W}n)$. Define

$$\mathring{A}_3(s, x, n, \mathsf{i}) := t.$$

Similarly define $\mathring{A}_3(s, x, n, 1)$ such that the corresponding site belongs to $R_{0,18(n+1)}$.

Algorithm 4

1) Set $f_{0,1} = \mathring{A}_3(s, x, n, \mathsf{i})$;
2) For $2 \le j \le m$
   set $f_{0,j} = \mathring{A}_3(f_{0,j-1}, n, \mathsf{i})$;
   End
3) Set $f_{j,0} = \infty$, $j = 1 \dots m$;
4) For $1 \le i \le m$
   For $1 \le j \le m$
   set $f_{i,j} = \mathring{A}_3(f_{i-1,j}, n, 1)1_{\{f_{i-1,j} < \infty\}} + \mathring{A}_3(f_{i,j-1}, n, \mathsf{i})1_{\{f_{i-1,j} = \infty\}}$;
   End
   End
5) Return $f_{m,m}$.

If $f_{m,m} < \infty$, then the corresponding site belongs to $R_{18(n+1)m,18(n+1)m}$. Define

$$\mathring{A}_4(s, x, n, m, 1 + \mathsf{i}) = f_{m,m}.$$

Similarly, define $\mathring{A}_4(s,x,n,m,o)$ for $o \in \{1-\mathrm{i}, -1+\mathrm{i}, -1-\mathrm{i}\}$.

Proposition 5.1 *Suppose* $p<1$ *and* $\mathbf{P}^p_\lambda(\xi^0$ *survives*$)>0$. *Let* $x = x(\varepsilon) \in \mathbb{H}$ *with* $\Im(x) > 10h$ *and* $(x\times 0)_r$ *be a horizontal seed. Then*

$$\lim_{\varepsilon\to 0+}\liminf_{n\to\infty}\liminf_{m\to\infty}\mathbf{P}^p_\lambda(200\overline{W}nm < \mathring{A}_4(0,x,n,m,1+\mathrm{i}) < 200\overline{W}n(m+1)) = 1.$$

Proof By Proposition 4.1 and the FKG inequality, we have that with large probability $\{\mathring{A}_3(s,x,n,\mathrm{i}), \mathring{A}_3(s,x,n,1)\} \subset [100k\overline{W}n, 100(k+1)\overline{W}n)$ if $s \in [100(k-1)\overline{W}n, 100k\overline{W}n)$. Such $(f_{i,j})$ corresponds to a 1-dependent site percolation. Using the result of 1-dependent site percolation (see Durrett [3]), we get the conclusion. □

Now we have finished the definition of $\mathring{A}_4$ and we can prove (`ii`). Suppose $(x\times 0)_r$ is a horizontal seed with $\Im(x) \geq 10h$. Let

$$\mu := \mathring{A}_4(\mathring{A}_4(\mathring{A}_4(\mathring{A}_4(0,x,n,m,1+\mathrm{i}),n,m,-1+\mathrm{i}),n,m,-1-\mathrm{i}),n,m-1,1-\mathrm{i})$$

and $(\mu\times\nu)_r$ be the corresponding seed if $\mu<\infty$. Then $\nu \in B_x(40n\mathrm{M}h)$. By Proposition 5.1 and the strong Markov property,

$$\lim_{\varepsilon\to 0+}\liminf_{n\to\infty}\liminf_{m\to\infty}\mathbf{P}^p_\lambda(800\overline{W}nm - 200\overline{W}n \leq \mu \leq 800\overline{W}nm + 600\overline{W}n) = 1.$$

That is,

$$\lim_{\varepsilon\to 0+}\liminf_{n\to\infty}\liminf_{m\to\infty}\mathbf{P}^p_\lambda(\exists t \in [800\overline{W}n(m-1), 800\overline{W}n(m+1)],$$
$$\xi_t^{\lceil x-r,x+r\rfloor} \cap B_x(40n\mathrm{M}h) \neq \emptyset) = 1.$$

We can deduce that for $\delta>0$ there exist ε, n and m_0, such that for all $m \geq m_0 \geq 2$,

$$\mathbf{P}^p_\lambda(\exists t \in [800\overline{W}n(m-1), 800\overline{W}n(m+1)],\ \xi_t^{\lceil x-r,x+r\rfloor} \cap B_x(40n\mathrm{M}h) \neq \emptyset) > 1-\delta.$$

On the other hand, consider Richardson's process (ζ_t^A) on $\mathbb{H}$ with parameter λ (see Richardson [12]). Then (ζ_t^A) stochastically dominates $\xi_t^A(\mathcal{C})$ for every $\mathcal{C}$. It is well known that there exists $\alpha_l>0$ with $\alpha_l\to 0$, such that for all $l > 40n\mathrm{M}h$ and $A \subseteq \mathbb{H}\setminus B_0(l)$,

$$\mathbf{P}(\exists 0<t<800\overline{W}n(m_0+1)+1,\ \zeta_t^A \cap B_x(40n\mathrm{M}h) \neq \emptyset) \leq \alpha_l.$$

Choose l_0 large enough so that $\alpha_{l_0} < \delta$. Set

$$\tau := \inf\{u \geq s-1 : \xi_u^{\lceil x-r,x+r\rfloor} \cap B_0(l_0) = \emptyset\}.$$

Then τ is a stopping time. For any $A \subseteq \mathbb{H}\setminus B_0(l_0)$,

$$\begin{aligned}
&\mathbf{P}_\lambda(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t+s}^{\lceil x-r,x+r\rfloor}(\mathcal{C}) \cap B_x(40n\mathrm{M}h) \neq \emptyset | \xi_\tau^{\lceil x-r,x+r\rfloor}(\mathcal{C}) = A)\\
&\leq \mathbf{P}(\exists 0<t<800\overline{W}n(m_0+1)+1,\ \zeta_t^A \cap B_x(40n\mathrm{M}h) \neq \emptyset)\\
&\leq \delta.
\end{aligned}$$

Use the strong Markov property again,

$$\begin{aligned}
&\mathbf{P}_\lambda(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t+s}^{\lceil x-r,x+r\rfloor}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset,\ \ \xi_u^{\lceil x-r,x+r\rfloor}\cap B_0(l_0)=\emptyset\\
&\quad\text{for some } u\in[s-1,s])\\
&\quad=\mathbf{P}_\lambda(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t+s}^{\lceil x-r,x+r\rfloor}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset,\tau\le s)\\
&\quad=\mathbf{P}_\lambda(\mathbf{P}_\lambda(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t}^{\lceil x-r,x+r\rfloor}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset|\mathcal{F}_\tau);\ \tau\le s)\\
&\quad=\mathbf{P}_\lambda(\mathbf{P}_\lambda(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t}^{\lceil x-r,x+r\rfloor}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset|\xi_\tau^{\lceil x-r,x+r\rfloor}(\mathcal{C}));\\
&\qquad\quad\tau\le s)\\
&\quad\le\delta\mathbf{P}_\lambda(\tau\le s)
\end{aligned}$$

for $s\ge 1$. That is,

$$\begin{aligned}
&\mathbf{P}_\lambda(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t+s}^{\lceil x-r,x+r\rfloor}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset\mid\xi_u^{\lceil x-r,x+r\rfloor}\cap B_0(l_0)=\emptyset\\
&\quad\text{for some } u\in[s-1,s])\le\delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{P}_\lambda^p(\xi_u^{\lceil x-r,x+r\rfloor}\cap B_0(l_0)\neq\emptyset \text{ for all } u\in[s-1,s])\\
&\quad\ge\mathbf{P}_\lambda^p(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t+s}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset)\\
&\qquad-\mathbf{P}_\lambda^p(\exists 0<t<800\overline{W}n(m_0+1),\ \xi_{t+s}(\mathcal{C})\cap B_x(40n\mathrm{M}h)\neq\emptyset|\ \xi_u^{\lceil x-r,x+r\rfloor}\cap B_0(l_0)=\emptyset\\
&\qquad\qquad\text{for some } u\in[s-1,s])\\
&\quad\ge 1-2\delta.
\end{aligned}$$

Since $\xi_t^{A_1}\subseteq\xi_t^{A_2}$ if $A_1\subseteq A_2$, we have

$$\mathbf{P}_\lambda^p(\xi_s^{B_0(l)}\cap B_0(l)\neq\emptyset \text{ for } s\in[t,t+1])\ge 1-2\delta$$

for any $l>l_0$ and $t>0$. Therefore, for any $n>0$, there exists l_n such that

$$\mathbf{P}_\lambda^p(\xi_s^{B_0(l_n)}\cap B_0(l_n)\neq\emptyset \text{ for } s\in[t,t+1])\ge 1-4^{-n}.$$

Let

$$\Omega_{n,t}:=\{\omega:\ \mathbf{P}_\lambda(\xi_s^{B_0(l)}\cap B_0(l)\neq\emptyset \text{ for } s\in[t,t+1])\ge 1-2^{-n}\}.$$

Then $\mathbf{P}^p(\Omega_{n,t})\ge 1-2^{-n}$ for all n and t. Let

$$\Omega_t:=\liminf_{n\to\infty}\Omega_{n,t}.$$

Then $\mathbf{P}^p(\Omega_t)=1$ for all t. Furthermore, let

$$\Omega_0''':=\bigcap_{k=1}^{\infty}\Omega_k.$$

Then $\mathbf{P}^p(\Omega_0''')=1$, and (ii) holds for all $\omega\in\Omega_0'''$.

Finally, set $\Omega_0:=\Omega_0''\cap\Omega_0'''$. As a result, (i) and (ii) hold for all $\omega\in\Omega_0$.

6 Concluding Remarks

Although we only study contact processes on open clusters of half space, our result contains a weaker result for contact processes on open clusters of whole space. Recall p^* the critical value of the Bernoulli bond percolation model on $\mathbb{Z}^d \times \mathbb{Z}^+$ in Sect. 1. Fix $p > p^*$. Let $\lambda_c^H := \inf\{\lambda : \mathbf{P}_\lambda^p(\xi^0 \text{ survives}) > 0\}$. Let $\mathcal{D}$ be open clusters generated by Bernoulli bond percolation of $\mathbb{Z}^{d+1}$ with parameter p. Obviously, $\lambda_c(\mathcal{D}) \le \lambda_c^H$ almost surely.

Theorem 6.1 *For $\lambda > \lambda_c^H$, the complete convergence theorem holds for contact processes with parameter λ on almost all $\mathcal{D}$.*

Proof It is directly drawn from Theorem 1.1 and Theorem 1.12 of Liggett [10]. □

When $p = 1$, it is well known that $\lambda_c(\mathcal{D}) = \lambda_c^H$, see [1]. Hence we have the following problem, which we believe true. If it is true, then the complete convergence theorem holds for supercritical contact process on almost all open clusters of whole space.

Problem 1 For $p^* < p < 1$, does $\lambda_c(\mathcal{D}) = \lambda_c^H$ almost surely?

It is a pity that our method does not work for the whole plane. In the half plane case, infected sites on the real line cannot infect any sites below. Together with the fact that the chance of the existence of crossing from bottom to top of a box is small if the box is high enough, we can get that the seed on the bottom, which is just on the real line, infects sufficiently many sites on the left and the right sides almost surely if the process survives. As a result, we can successfully use a seed to generate other two seeds and construct the two restart processes which are independent of the former process in the annealed law. The rest proof is routine, see [1, 6]. However, our method fails in the whole plane case. On whole plane, infected sites on the real line can infect sites below. We cannot ensure that a seed on the bottom of a box can infect sufficiently many sites on the left and right sides with high probability, so that, we cannot find independent restart processes.

In this paper, we show that the complete convergence theorem holds for contact processes on most subgraphs of $\mathbb{Z}^d \times \mathbb{Z}^+$. But whether it is true for all subgraphs, which is more interesting, is still unknown. Hence we propose the following problem.

Problem 2 Does the complete convergence theorem hold for any infinite connected graph G which can be embedded in $\mathbb{Z}^d \times \mathbb{Z}^+$?

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Appendix 1: Proof of Lemmas 3.5 and 3.6

Proof of Lemma 3.5 Fix $\omega \in \Omega^b$, $w \in \lceil 4h + 4r, [4.0001h]\rfloor$ and $a \in \lceil 0, h\mathrm{i}\rfloor$. Set $s = x + w + a\mathrm{i}$. Define $(\mathcal{M}_t, t \ge 0)$ to be a filtration:

$$\mathcal{M}_t := \sigma\left(\xi_u^{\lceil x-r, x+r\rfloor, 0, \langle x-4h-1, x+w+h\mathrm{i}\rangle \cup (s+\mathbb{G})} : 0 \le u \le t\right).$$

Then $T' = T + \infty \cdot 1_{\{S \neq s\}}$ is a $\mathcal{M}_t$-stopping time, $\tilde{\xi}_t^{s,\mathbb{G}} \in \mathcal{M}_t$ and $T \cdot 1_{\{T<\infty,\ S=s\}} \in \mathcal{M}_{T'}$. By the strong Markov property,

$$\begin{aligned}
&\mathbf{P}_\lambda(T < \infty, S = s, T \in B,\ \ \tilde{\xi}^{S,\mathbb{G}} \circ \theta_T \in C) \\
&\quad = \mathbf{E}_\lambda(\mathbf{P}_\lambda(T < \infty, S = s, T \in B,\ \tilde{\xi}^{S,\mathbb{G}} \circ \theta_T \in C | \mathcal{M}_{T'})) \\
&\quad = \mathbf{E}_\lambda(\mathbf{P}_\lambda(\tilde{\xi}^{s,\mathbb{G}} \circ \theta_{T'} \in C | \mathcal{M}_{T'}); T \in B, T < \infty, S = s) \\
&\quad = \mathbf{P}_\lambda(\tilde{\xi}^{s,\mathbb{G}} \in C)\mathbf{P}_\lambda(T \in B, T < \infty, S = s),
\end{aligned}$$

where $\mathbf{E}_\lambda$ is the expectation corresponding to $\mathbf{P}_\lambda$. The assumption that $s + \mathbb{G}$ and $\langle x - 4h - 1, x + w + h\mathrm{i}\rangle$ shares no edges implies

$$\mathbf{E}^p(\mathbf{P}_\lambda(\tilde{\xi}^{s,\mathbb{G}} \in C)\mathbf{P}_\lambda(T \in B, T < \infty, S = s)) = \mathbf{P}_\lambda^p(\tilde{\xi}^{s,\mathbb{G}} \in C)\mathbf{P}_\lambda^p(T \in B, T < \infty, S = s).$$

By the property of translation invariance,

$$\begin{aligned}
\mathbf{P}_\lambda^p(\tilde{\xi}^{s,\mathbb{G}} \in C) &= \mathbf{E}^p(\mathbf{P}_\lambda(\xi^{\lceil s - r\mathrm{i}, s + r\mathrm{i}\rfloor, 0, s+\mathbb{G}}(\mathcal{C}) - s \in C)) \\
&= \mathbf{E}^p(\mathbf{P}_\lambda(\xi^{\lceil x - r\mathrm{i}, x + r\mathrm{i}\rfloor, 0, x+\mathbb{G}}(\mathcal{C} - s + x) - x \in C)) \\
&= \mathbf{E}^p(\mathbf{P}_\lambda(\xi^{\lceil x - r\mathrm{i}, x + r\mathrm{i}\rfloor, 0, x+\mathbb{G}}(\mathcal{C}) - x \in C)) \\
&= \mathbf{P}_\lambda^p(\tilde{\xi}^{x,\mathbb{G}} \in C),
\end{aligned}$$

where $\mathbf{E}^p$ is the expectation corresponding to $\mathbf{P}^p$. Therefore,

$$\mathbf{P}_\lambda^p(T < \infty,\ T \in B,\ S = s,\ \tilde{\xi}^{S,\mathbb{G}} \circ \theta_T \in C) = \mathbf{P}_\lambda^p(T < \infty,\ T \in B,\ S = s,)\mathbf{P}_\lambda^p(\tilde{\xi}^{x,\mathbb{G}} \in C),$$

as desired. $\square$

Proof of Lemma 3.6 Fix $\omega \in \Omega^b$, $\{w_1, w_{-1}\} \subset \lceil 8h + 4r, [8.0001h]\rfloor$ and $\{a_1, a_{-1}\} \subset \lceil 0, h\mathrm{i}\rfloor$. Set $s_1 = x + w_1 + a_1\mathrm{i}$ and $s_{-1} = x - w_{-1} + a_{-1}\mathrm{i}$. Define $(\mathcal{M}_t, t \geq 0)$ to be a filtration:

$$\mathcal{M}_t := \sigma\big(\xi_u^{\lceil x-r, x+r\rfloor, 0, \langle x - w_{-1}, x + w_1 + h\mathrm{i}\rangle \cup (s_1 + \mathbb{G}_1) \cup (s_{-1} + \mathbb{G}_{-1})} : 0 \leq u \leq t\big).$$

Then $T_1' = T_1 + \infty \cdot 1_{\{S_1 \neq s_1\}}$ and $T_{-1}' = T_{-1} + \infty \cdot 1_{\{S_{-1} \neq s_{-1}\}}$ are $\mathcal{M}_t$-stopping time. Directly calculate

$$\begin{aligned}
&\mathbf{P}_\lambda(S_1 = s_1,\ \ T_1 \in B_1 \cap [0, \infty),\ \tilde{\xi}^{S_1, \mathbb{G}_1} \circ \theta_{T_1} \in C_1, \\
&\qquad S_{-1} = s_{-1},\ \ T_{-1} \in B_{-1} \cap [0, \infty),\ \tilde{\xi}^{S_{-1}, \mathbb{G}_{-1}} \circ \theta_{T_{-1}} \in C_{-1}, T_1 < T_{-1}) \\
&\quad = \mathbf{E}_\lambda(\mathbf{P}_\lambda(S_1 = s_1,\ \ T_1 \in B_1 \cap [0, \infty),\ \tilde{\xi}^{s_1, \mathbb{G}_1} \circ \theta_{T_1} \in C_1, \\
&\qquad\quad S_{-1} = s_{-1},\ \ T_{-1} \in B_{-1} \cap [0, \infty),\ \tilde{\xi}^{S_{-1}, \mathbb{G}_{-1}} \circ \theta_{T_{-1}} \in C_{-1}, T_1' < T_{-1}' | \mathcal{M}_{T_1'})) \\
&\quad = \mathbf{E}_\lambda(\mathbf{P}_\lambda(S_{-1} = s_{-1},\ T_{-1} \in B_{-1} \cap [0, \infty),\ \tilde{\xi}^{s_1, \mathbb{G}_1} \circ \theta_{T_1} \in C_1, \\
&\qquad\quad \tilde{\xi}^{S_{-1}, \mathbb{G}_{-1}} \circ \theta_{T_{-1}} \in C_{-1} | \mathcal{M}_{T_1'}); S_1 = s_1, T_1 \in B_1 \cap [0, \infty), T_1 < T_{-1}') \\
&\quad = \mathbf{E}_\lambda(\mathbf{P}_\lambda^D(S_{-1} = s_{-1},\ \ T_{-1} \in B_{-1} \cap [0, \infty),\ \tilde{\xi}^{s_1, \mathbb{G}_1} \in C_1,\ \tilde{\xi}^{S_{-1}, \mathbb{G}_{-1}} \circ \theta_{T_{-1}} \in C_{-1}); \\
&\qquad\quad S_1 = s_1, T_1 \in B_1 \cap [0, \infty), T_1 < T_{-1})
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}_\lambda(\mathbf{P}_\lambda^{D_{-1}\cup\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor}(S_{-1}=s_{-1},\ \ T_{-1}\in B_{-1}\cap[0,\infty),\ \tilde{\xi}^{s_1,\mathbb{G}_1}\in C_1,\\
&\qquad \tilde{\xi}^{S_{-1},\mathbb{G}_{-1}}\circ\theta_{T_{-1}}\in C_{-1});\, S_1=s_1, T_1\in B_1\cap[0,\infty), T_1<T_{-1})\\
&= \mathbf{E}_\lambda(\mathbf{P}_\lambda^{D_{-1}}(S_{-1}=s_{-1},\ \ T_{-1}\in B_{-1}\cap[0,\infty),\ \tilde{\xi}^{S_{-1},\mathbb{G}_{-1}}\circ\theta_{T_{-1}}\in C_{-1})\\
&\qquad \times \mathbf{P}_\lambda^{\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor}(\tilde{\xi}^{s_1,\mathbb{G}_1}\in C_1);\, S_1=s_1, T_1\in B_1\cap[0,\infty), T_1<T_{-1})\\
&= \mathbf{E}_\lambda(\mathbf{P}_\lambda(S_{-1}=s_{-1},\ \ T_{-1}\in B_{-1}\cap[0,\infty),\ \tilde{\xi}^{S_{-1},\mathbb{G}_{-1}}\circ\theta_{T_{-1}}\in C_{-1}|\mathcal{M}_{T_1'});\\
&\qquad S_1=s_1, T_1\in B_1\cap[0,\infty), T_1<T_{-1})\mathbf{P}_\lambda^{\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor}(\tilde{\xi}^{s_1,\mathbb{G}_1}\in C_1)\\
&= \mathbf{P}_\lambda(S_{-1}=s_{-1},\ \ T_{-1}\in B_{-1}\cap[0,\infty),\ \tilde{\xi}^{S_{-1},\mathbb{G}_{-1}}\circ\theta_{T_{-1}}\in C_{-1},\\
&\qquad S_1=s_1, T_1\in B_1\cap[0,\infty), T_1<T_{-1})\mathbf{P}_\lambda^{\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor}(\tilde{\xi}^{s_1,\mathbb{G}_1}\in C_1)\\
&= \mathbf{E}_\lambda(\mathbf{P}_\lambda(S_{-1}=s_{-1},\ \ T_{-1}\in B_{-1}\cap[0,\infty),\ \tilde{\xi}^{S_{-1},\mathbb{G}_{-1}}\circ\theta_{T_{-1}}\in C_{-1},\\
&\qquad = S_1 s_1, T_1\in B_1\cap[0,\infty), T_1<T_{-1}|\mathcal{M}_{T'_{-1}}))\mathbf{P}_\lambda^{\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor}(\tilde{\xi}^{s_1,\mathbb{G}_1}\in C_1)\\
&= \mathbf{P}_\lambda(S_{-1}=s_{-1},\ \ T_{-1}\in B_{-1}\cap[0,\infty), S_1=s_1, T_1\in B_1\cap[0,\infty), T_1<T_{-1})\\
&\quad \times \mathbf{P}_\lambda^{\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor}(\tilde{\xi}^{s_1,\mathbb{G}_1}\in C_1)\mathbf{P}_\lambda^{\lceil s_{-1}-r\mathrm{i},s_{-1}+r\mathrm{i}\rfloor}(\tilde{\xi}^{s_{-1},\mathbb{G}_{-1}}\in C_{-1}),
\end{aligned}
$$

where $\mathbf{P}_\lambda^A = \mathbf{P}_\lambda \circ (\xi^A)^{-1}$ is the probability measure induced by ξ^A for $A \subseteq \mathbb{H}$ and

$$
D = \xi_{T_1}^{\lceil -r,r\rfloor,0,\langle x-w_{-1},x+w_1+h\mathrm{i}\rangle\cup(s_1+\mathbb{G}_1)\cup(s_{-1}+\mathbb{G}_{-1})}, \qquad D_{-1} = \xi_{T_1}^{\lceil -r,r\rfloor,0,\langle x-w_{-1},x+8h+h\mathrm{i}\rangle};
$$

the third and the last equalities are based on the strong Markov property; the fourth equality holds since

$$
\xi^D = \xi^{D_{-1}} \cup \xi^{\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor} \cup \xi^{D\setminus(D_{-1}\cup\lceil s_1-r\mathrm{i},s_1+r\mathrm{i}\rfloor)};
$$

and the fifth equality holds since $(s_1+\mathbb{G}_1)$ shares no endpoints with $(s_{-1}+\mathbb{G}_{-1}) \cup \langle x-w_{-1}, x+8h+1+h\mathrm{i}\rangle$.

We omit the rest of the proof since it is parallel to that of Lemma 3.5. □

Appendix 2: Proof of Proposition 4.1

We need more preparations. By our construction and the symmetric property, $\mathring{A}_L(0,x,n,1+\mathrm{i})$ and $\mathring{A}_R(0,x,n,1+\mathrm{i})$ have the same distribution. So we only need to prove

$$
\lim_{\varepsilon\to 0+}\liminf_{n\to\infty}\mathbf{P}_\lambda^p\left(\frac{7\overline{W}}{6}n < \mathring{A}_L(0,x,n,1+\mathrm{i}) < \frac{11\overline{W}}{6}n\right) = 1.
$$

We shall introduce a random variable $s_{n,n}^{\mathrm{i}}$. It has the same distribution with $\mathring{A}_L(0,x,n,1+\mathrm{i})$ and is the sum of a sequence of independent and identically distributed random variables with finite means conditioned on some large probability event. The random variable $s_{n,n}^{\mathrm{i}}$ is constructed by Algorithms 5 and 6, which are similar to Algorithms 1 and 2. Define random variables SB_k, LB_k, TB_k and random vectors $S_k=(S_k(1),S_k(2))$, $L_k=(L_k(1),L_k(2))$, $T_k=(T_k(1),T_k(2),T_k(3),T_k(4))$ for $k\ge 0$ as follows:

$$\mathbf{P}(SB_k=1)=1-\mathbf{P}(SB_k=0)=\mathbf{P}^p_\lambda(S_T(0,0,1,1)<\infty);$$
$$\mathbf{P}(LB_k=1)=1-\mathbf{P}(LB_k=0)=\mathbf{P}^p_\lambda(L_T(0,0,1,1)<\infty);$$
$$\mathbf{P}(TB_k=1)=1-\mathbf{P}(TB_k=0)=\mathbf{P}^p_\lambda(L_T(0,0,1,1)+L_T(0,0,1,-1)<\infty);$$
$$\mathbf{P}(S_k\in\cdot\,)=\mathbf{P}^p_\lambda((S_T(0,0,1,1),S_S(0,0,1,1))\in\cdot\mid S_T(0,0,1,1)<\infty);$$
$$\mathbf{P}(L_k\in\cdot\,)=\mathbf{P}^p_\lambda((L_T(0,0,1,1),L_S(0,0,1,1))\in\cdot\mid L_T(0,0,1,1)<\infty);$$
$$\mathbf{P}(T_k\in\cdot\,)=\mathbf{P}^p_\lambda((L_T(0,0,1,1),L_T(0,0,1,-1),L_S(0,0,1,1),L_S(0,0,1,-1)) \in\cdot\mid L_T(0,0,1,1)+L_T(0,0,1,-1)<\infty).$$

Let all $SB_k, LB_k, TB_k, S_k, L_k$ and T_k for $k\ge 0$ be independent.

Now give $j_0, k_0, n_0\in\mathbb{Z}^+$, then we have the following algorithm.

Algorithm 5

0) Set $j=j_0, k=k_0, (t,y)=(s,x)$ and $t_1=t_2=\infty$.
1) If $SB_j=1$, then set $(t,y)=(S_j(1)+t, e^{\mathrm{i}\frac{\pi}{2}}S_j(2)+y)$ and $j=j+1$. Otherwise go to 17).
2) If $SB_j=1$, then set $(t,y)=(S_j(1)+t,-\overline{S_j(2)}+y)$ and $j=j+1$. Otherwise go to 17).
3) If $y\in\lceil a+\mathrm{M}h\mathrm{i}+50h, a+\mathrm{M}h\mathrm{i}+60h+90h\mathrm{i}\rfloor$, then go to 14).
4) If $\Re(y)\ge\Re(a)+30h$, then go to 1).
5) If $LB_k=1$, then set $(t,y)=(L_k(1)+t, L_k(2)e^{\mathrm{i}\frac{\pi}{2}}+y)$ and $k=k+1$. Otherwise go to 17).
6) If $LB_k=1$, then set $(t,y)=(L_k(1)+t, L_k(2)+y)$ and $k=k+1$. Otherwise go to 17).
7) If $SB_j=1$, then set $(t,y)=(S_j(1)+t,\overline{S_j(2)}e^{\mathrm{i}\frac{\pi}{2}}+y)$ and $j=j+1$. Otherwise go to 17).
8) If $SB_j=1$, then set $(t,y)=(S_j(1)+t, S_j(2)+y)$ and $j=j+1$. Otherwise go to 17).
9) If $y\in\lceil a+\mathrm{M}h\mathrm{i}+40h, a+\mathrm{M}h\mathrm{i}+50h+90h\mathrm{i}\rfloor$, then go to 15).
10) If $\Re(y)\le\Re(a)+70h$, then go to 7).
11) If $LB_k=1$, then set $(t,y)=(L_k(1)+t,\overline{L_k(2)}e^{\mathrm{i}\frac{\pi}{2}}+y)$ and $k=k+1$. Otherwise go to 17).
12) If $LB_k=1$, then set $(t,y)=(L_k(1)+t,-\overline{L_k(2)}+y)$ and $k=k+1$. Otherwise go to 17).
13) Go to 1).
14) If $LB_k=1$, set $(t,y)=(L_k(1)+t, L_k(2)e^{\mathrm{i}\frac{\pi}{2}}+y)$, $k=k+1$ and go to 16). Otherwise go to 17).
15) If $LB_k=1$, set $(t,y)=(L_k(1)+t,\overline{L_k(2)}e^{\mathrm{i}\frac{\pi}{2}}+y)$ and $k=k+1$. Otherwise go to 17).
16) If $TB_{n_0}=1$, then set $(t_1,t_2,y_1,y_2)=T_{n_0}+(t,t,y,y)$.
17) Return t_1,t_2,y_1,y_2,j,k.

If $t_1+t_2<\infty$, we crudely calculate and get $k_{lower}\le k-k_0\le k_{upper}$, $j_{lower}\le j-j_0\le j_{upper}$, where

$$\begin{aligned} k_{upper}&=\frac{5.0001\times(\mathrm{M}+200)}{80}, & k_{lower}&=\frac{\mathrm{M}}{5\times 5.0001},\\ j_{upper}&=\frac{\mathrm{M}+200-4k_{lower}}{2}, & j_{lower}&=\frac{2\mathrm{M}-9.0001k_{upper}}{5.0001}. \end{aligned} \tag{C.1}$$

Our choice of $\mathrm{M} = 10^7$ ensures that $\frac{k_{upper}}{k_{lower}} < \frac{11}{7}$, $\frac{j_{upper}}{j_{lower}} < \frac{11}{7}$. Define $\mathring{A}_5^v(s, x, j_0, k_0, n_0, \mathrm{i}) = (t_1, t_2, y_1, y_2, j, k)$. By Lemmas 3.5 and 3.6, one can get that (t_1, t_2, y_1, y_2) and $\mathring{A}_1^v(s, x, \mathrm{i})$ have the same distribution.

Similarly, we omit the space parameters and abuse the notation $\mathring{A}_5^v(s, j_0, k_0, n_0, o) = (t_1, t_2, j, k)$. As before, we use $\mathring{A}_5$ to stand for $\mathring{A}_5^v$ and $\mathring{A}_5^h$ without distinguishing.

Algorithm 6

1) Set $(s^{\mathrm{i}}_{0,1}, s^1_{0,1}, j_{0,1}, k_{0,1}) = \mathring{A}_5(0, x, 0, 0, 0, \mathrm{i})$, $f = 0$, $U_{0,0} = 0$;
2) For $2 \le l \le n$
    set $f = f + 1$;
    set $U_{0,l} = f$;
    set $(s^{\mathrm{i}}_{0,l}, s^1_{0,l}, j_{0,l}, k_{0,l}) = \mathring{A}_5(s^{\mathrm{i}}_{0,l-1}, \mathrm{M}f, \mathrm{M}f, \mathrm{M}f, \mathrm{i})$;
  End
3) Set $t^{\mathrm{i}}_{j,0} = \infty, l = 1, \dots, n$
4) For $1 \le i \le n$
    For $1 \le l \le n$
      set $f = f + 1$;
      set $U_{i,l} = f$;
      set $(s^{\mathrm{i}}_{i,l}, s^1_{i,l}, j_{i,l}, k_{i,l}) = \mathring{A}_5(s^1_{i-1,l}, \mathrm{M}f, \mathrm{M}f, \mathrm{M}f, 1)1_{\{s^1_{i-1,l}<\infty\}}$
        $+ \mathring{A}_5(s^{\mathrm{i}}_{i,l-1}, \mathrm{M}f, \mathrm{M}f, \mathrm{M}f, \mathrm{i})1_{\{s^1_{i-1,l}=\infty\}}$;
      End
    End
5) Return $(s^{\mathrm{i}}_{i,l}, s^1_{i,l}, j_{i,l}, k_{i,l}, U_{i,l} : 1 \le i \le n, 1 \le l \le n)$.

By Lemmas 3.5 and 3.6, $s^{\mathrm{i}}_{n,n}$ has the same distribution as $\mathring{A}_L(0, x, n, 1 + \mathrm{i})$. The matrices $(U_{i,l})$ are determined, which will be used later. Before the proof of Proposition 4.1 we need the following lemma.

Lemma A.1 Let $X, X_1, X_2, \dots$ be independent and identically distributed random elements. Let $\mathcal{F}_n$ be a σ-field, $\mathcal{F}_n \supseteq \sigma\{X_1, \dots, X_n\}$ and X_{n+1} is independent of $\mathcal{F}_n$. Let T be an $\mathcal{F}_n$-stopping time and $T < M$, where M is a fixed positive integer. Let $f(x)$ be a function satisfying $f(x) > x$ for all x. Then $(X_1, \dots, X_M)$ has the same distribution as $(X_1, \dots, X_T, X_{f(T)}, \dots, X_{f(T)+M-T-1})$.

Proof For any sets $A_1, \dots, A_M \in \sigma(X)$,

$$
\begin{aligned}
&\mathbf{P}(X_1 \in A_1, \dots, X_T \in A_T, X_{f(T)} \in A_{T+1}, \dots, X_{f(T)+M-T-1} \in A_M) \\
&\quad = \mathbf{E}(\mathbf{P}(X_1 \in A_1, \dots, X_T \in A_T, X_{f(T)} \in A_{T+1}, \dots, X_{f(T)+M-T-1} \in A_M | \mathcal{F}_T)) \\
&\quad = \mathbf{E}(\mathbf{P}(X_{f(T)} \in A_{T+1}, \dots, X_{f(T)+M-T-1} \in A_M | \mathcal{F}_T); X_1 \in A_1, \dots, X_T \in A_T) \\
&\quad = \mathbf{E}\left(\prod_{i=T+1}^{M} \mathbf{P}(X \in A_i); X_1 \in A_1, \dots, X_T \in A_T\right) \\
&\quad = \mathbf{E}(\mathbf{P}(X_{T+1} \in A_{T+1}, \dots, X_M \in A_M | \mathcal{F}_T); X_1 \in A_1, \dots, X_T \in A_T) \\
&\quad = \mathbf{P}(X_1 \in A_1, \dots, X_M \in A_M).
\end{aligned}
$$

We complete the proof of the lemma. □

Proof of Proposition 4.1 Since $\mathring{A}_L(0, x, n, 1+\mathrm{i})$ has the same distribution as $s^{\mathrm{i}}_{n,n}$, we need only estimate $s^{\mathrm{i}}_{n,n}$. Generate a labeled graph $\mathcal{C}'$ of $\mathbb{H}$ in the following way: $i + j\mathrm{i}$ is an open site of $\mathcal{C}'$ if and only if $SB_k = LB_k = TB_k = 1$ for all $\mathrm{M}U_{i,j} \le k < \mathrm{M}(U_{i,j}+1)$. We say that $a_1 \sim \cdots \sim a_n$ is an open path of $\mathcal{C}'$ if $a_i \in \mathcal{C}'$ and $a_{i+1} - a_i \in \{1, \mathrm{i}\}$. Set A_n be the event that there exists an open path from $(0,0)$ to (n,n) in $\mathcal{C}'$. Obviously, if A_n occurs then $s^{\mathrm{i}}_{n,n} < \infty$. Moreover, on A_n, there exists a unique open path $0 = P_0 \sim \cdots \sim P_m \sim \cdots \sim P_{2n} = n + n\mathrm{i}$ which satisfies

$$(s^{\mathrm{i}}_{P_m}, s^{1}_{P_m}, j_{P_m}, k_{P_m}) = \mathring{A}_5(s^{P_m - P_{m-1}}_{P_{m-1}}, \mathrm{M}U_{P_m}, \mathrm{M}U_{P_m}, \mathrm{M}U_{P_m}, P_m - P_{m-1})$$

for $0 \le m \le 2n$. Here we set $P_{-1} = -\mathrm{i}$, and do not distinguish P_m with (i, j) if $P_m = i + j\mathrm{i}$. Furthermore,

$$s^{\mathrm{i}}_{n,n} = \sum_{m=0}^{2n} \left(\sum_{h=\mathrm{M}U_{P_m}}^{j_{P_m}-1} S_h(1) + \sum_{h=\mathrm{M}U_{P_m}}^{k_{P_m}-1} L_h(1) + T_{\mathrm{M}U_{P_m}}(*) \right),$$

where $*$ determined by (P_m) is equal to 1 or 2. Actually, $P_m 1_{A_n}$ is a measurable variable with respect to $\mathcal{E}_\infty = \sigma\{SB_m, LB_m, TB_m : m \ge 0\}$. Let $\mathcal{S}_m := \sigma\{S_h : 0 \le h \le m\}$, $\mathcal{L}_m := \sigma\{L_h : 0 \le h \le m\}$ and $\mathcal{T}_m := \sigma\{T_h : 0 \le h \le m\}$. Then $j_{P_m} - 1$ is an $\mathcal{H}_h$-stopping time conditioned on $\mathcal{E}_\infty$, $\mathcal{L}_\infty$ and $\mathcal{T}_\infty$. Let

$$(\hat{S}_1, \hat{S}_2, \ldots) := (S_{\mathrm{M}U_{P_0}}, \ldots, S_{j_{P_0}-1}, S_{\mathrm{M}U_{P_1}}, \ldots, S_{j_{P_1}-1}, \ldots, S_{\mathrm{M}U_{P_{2n}}}, \ldots, S_{j_{P_{2n}}-1}, S_{j_{P_{2n}}}, \\ \cdots | \mathcal{E}_\infty,\ \mathcal{L}_\infty,\ \mathcal{T}_\infty).$$

Then by Lemmas 3.5, 3.6 and our construction ensuring that all S-boxes are disjoint, $(\hat{S}_1, \hat{S}_2, \ldots)$ has the same distribution with $(S_1, S_2, \ldots)$. Refer to Fig. 9. Now on A_n, we write $a = \sum_{m=0}^{2n} (j_{P_m} - \mathrm{M}U_{P_m})$. By (A.1), there exist a_1 and a_2 which depend on M only, such that $a_1 n < a < a_2 n$ with $\frac{a_2}{a_1} < \frac{11}{7}$. As a result,

$$\begin{aligned}
&\mathbf{P}\left(\sum_{m=0}^{2n} \sum_{h=\mathrm{M}U_{P_m}}^{j_{P_m}-1} S_h(1) > \frac{11}{6}\overline{S}n \text{ or } \sum_{m=0}^{2n} \sum_{h=\mathrm{M}U_{P_m}}^{j_{P_m}-1} S_h(1) < \frac{7}{6}\overline{S}n;\, A_n \right) \\
&= \mathbf{E}\left(\mathbf{E}\left(\sum_{m=0}^{2n} \sum_{h=\mathrm{M}U_{P_m}}^{j_{P_m}-1} S_h(1) > \frac{11}{6}\overline{S}n \text{ or } \sum_{m=0}^{2n} \sum_{h=\mathrm{M}U_{P_m}}^{j_{P_m}-1} S_h(1) < \frac{7}{6}\overline{S}n | \mathcal{E}_\infty, \mathcal{L}_\infty, \mathcal{T}_\infty \right);\, A_n \right) \\
&\le \mathbf{E}\left(\mathbf{E}\left(\sum_{h=1}^{a_2 n} \hat{S}_h(1) > \frac{11}{6}\overline{S}n \ \text{ or } \ \sum_{h=1}^{a_1 n} \hat{S}_h(1) < \frac{7}{6}\overline{S}n \right);\, A_n \right) \\
&\le \mathbf{P}\left(\sum_{h=1}^{a_2 n} \hat{S}_h(1) > \frac{11}{6}\overline{S}n \right) + \mathbf{P}\left(\sum_{h=1}^{a_1 n} \hat{S}_h(1) < \frac{7}{6}\overline{S}n \right),
\end{aligned}$$

where $\overline{S} = 3(\frac{a_2}{11} + \frac{a_1}{7})\mathbf{E}(S_0(1))$. Furthermore, the probability above converges to zero as n goes to infinity by the central limit theorem. Similarly, there exist $\overline{L}$ and $\overline{T}$ which depend only on M, $\mathbf{E}(L_0(1))$ and $\mathbf{E}(T_0(1))$, such that

$$\mathbf{P}\left(\sum_{m=0}^{2n}\sum_{h=\mathrm{M}U_{P_m}}^{k_{P_m}-1} L_h(1) > \frac{11}{6}\overline{L}n \text{ or } \sum_{m=0}^{2n}\sum_{h=\mathrm{M}U_{P_m}}^{k_{P_m}-1} L_h(1) < \frac{7}{6}\overline{L}n\right) \to 0;$$

$$\mathbf{P}\left(\sum_{m=0}^{2n} T_{\mathrm{M}U_{P_m}}(*) > \frac{11}{6}\overline{T}n \text{ or } \sum_{m=0}^{2n} T_{\mathrm{M}U_{P_m}}(*) < \frac{7}{6}\overline{T}n\right) \to 0.$$

Finally, set $\overline{W} = \overline{S} + \overline{L} + \overline{T}$. Then $\overline{W} > 0$ and

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbf{P}\left(\frac{7}{6}\overline{W}n < s^{\mathrm{i}}_{n,n} < \frac{11}{6}\overline{W}n\right) \\
&\quad\geq \liminf_{n\to\infty} \mathbf{P}(A_n) - \lim_{n\to\infty} \mathbf{P}\left(s^{\mathrm{i}}_{n,n} > \frac{11}{6}\overline{W}n \text{ or } s^{\mathrm{i}}_{n,n} < \frac{7}{6}\overline{W}n;\, A_n\right) \\
&\quad= \liminf_{n\to\infty} \mathbf{P}(A_n).
\end{aligned}$$

But $\mathcal{C}'$ can be seen as an oriented site percolation with parameter $[\mathbf{P}(SB_1 = 1)]^{\mathrm{M}}[\mathbf{P}(LB_1 = 1)]^{\mathrm{M}}[\mathbf{P}(TB_1 = 1)]^{\mathrm{M}}$. By Lemma 3.4 we know that

$$\lim_{\varepsilon\to 0+} [\mathbf{P}(SB_1 = 1)]^{\mathrm{M}}[\mathbf{P}(LB_1 = 1)]^{\mathrm{M}}[\mathbf{P}(TB_1 = 1)]^{\mathrm{M}} = 1,$$

which implies that

$$\lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \mathbf{P}(A_n) = 1.$$

For more details about oriented site percolation one can refer to Durrett [3]. Therefore,

$$\lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \mathbf{P}\left(\frac{7}{6}\overline{W}n < s^{\mathrm{i}}_{n,n} < \frac{11}{6}\overline{W}n\right) = 1.$$

We have completed the proof of the proposition. □

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